Supplementary material for "Real option valuation of a decremental regulation service provided by electricity storage"

Dávid Zoltán Szabó[a], Randall Martyr[b]

I. A VERIFICATION THEOREM FOR THE STORAGE OPERATOR'S VALUATION PROBLEM

The aim of this section is to establish some basic theoretical results on the single option contract's value. In order to present the mathematical problem we must first recall some notation and make some technical assumptions. The instantaneous imbalance $X = (X(t))_{t\geq 0}$ is modelled as a Brownian motion with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_x)$, where for each $x \in \mathbb{R}$, $\mathbb{P}_x$ denotes a probability measure under which $\mathbb{P}_x(\{X(0) = x\}) = 1$. Let $\mathcal{T}$ denote the set of $\mathbb{F}$-stopping times, $r > 0$ represent a constant discount rate, and set $a = \sqrt{2r}$.

We frequently use the following well-known formula for the Laplace transform of the hitting time of X to a given point $y \in \mathbb{R}$, $D_{\{y\}} = \inf\{t \geq 0 : X(t) = y\}$ (see [1], for example):

$$\mathbb{E}_x\{e^{-rD_{\{y\}}}\} = e^{-a|y-x|}, \quad a = \sqrt{2r}.$$

We will also use the fact that X is a strong Markov process relative to $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_x)$, and assume furthermore that there exists a family of Markov shift operators $\theta_t : \Omega \to \Omega$, $t \geq 0$. For example, if $(\Omega, \mathcal{F})$ equals to the canonical space of continuous trajectories, $\Omega = C([0, \infty); \mathbb{R})$ equipped with its Borel σ-algebra $\mathcal{F} = \mathcal{B}(\Omega)$ (see Chapter 2 of [2]), then the shift operator θ_t is well defined by $\theta_t(\omega)(s) = \omega(t + s)$ for $\omega = (\omega(s))_{s\geq 0} \in \Omega$ and $t, s \geq 0$. Finally, we let $x \mapsto \psi_r(x) := e^{ax}$ and $x \mapsto \phi_r(x) := e^{-ax}$ denote the fundamental increasing and decreasing solutions of the differential equation $\frac{1}{2}\frac{d^2}{dx^2}w(x) = rw(x)$.

A. The storage operator's optimisation problem

In regime I, the storage operator faces an optimisation problem over two stopping times, τ_1 and τ_2, in which τ_1 represents the time for selling electricity, and τ_2 is the time of entry into the balancing services contract. If we define a set $\mathcal{T}_2$ by

$$\mathcal{T}_2 := \{(\tau_1, \tau_2) \in \mathcal{T} \times \mathcal{T} : \tau_1 \leq \tau_2\}, \tag{1}$$

this problem can be written mathematically by

$$V_I(x) = \sup_{(\tau_I,\tau_{II})\in\mathcal{T}_2} \mathbb{E}_x\{e^{-r\tau_I}(f(X(\tau_I))) + e^{-r\tau_{II}}(p + h_{III}(X(\tau_{II})))\}, \tag{2}$$

where f satisfies

$$f(x) = \begin{cases} M, & x \leq \frac{M-c}{b} \\ c + bx, & \frac{M-c}{b} < x \leq \frac{-c}{b} \\ 0, & \frac{-c}{b} < x \end{cases} \tag{3}$$

with constants M, c, b satisfying $M > c > 0$ and $b < 0$. This problem is a special case of the optimal starting–stopping problems studied in [3] (later extended in [4]). In this framework, we can analyse the balancing services contract's value by studying maximisation of the following functional:

$$J_x(\tau_1, \tau_2) = \mathbb{E}_x\Big\{L_1(X(\tau_1))e^{-r\tau_1} + L_2(X(\tau_2))e^{-r\tau_2}\Big\}, \ (\tau_1, \tau_2) \in \mathcal{T}_2 \tag{4}$$

[a] School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom. email: davidzoltan.szabo@postgrad.manchester.ac.uk

[b] School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom. email: r.martyr@qmul.ac.uk

where L_1 and L_2 are given real-valued functions, and the strategy space $\mathcal{T}_2$ was previously defined in (1). The value function $u\colon \mathbb{R} \to \mathbb{R}$ to this starting–stopping problem is defined by:

$$u(x) = \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} \mathbb{E}_x\{J_x(\tau_1,\tau_2)\}. \tag{5}$$

Remark I.1. The papers [3, 4] actually studied the functional (4) with an additional integral term:

$$J_x^L(\tau_1,\tau_2) = \mathbb{E}_x\Big\{\int_{\tau_1}^{\tau_2} e^{-rt}L(X(t))dt + L_1(X(\tau_1))e^{-r\tau_1} + L_2(X(\tau_2))e^{-r\tau_2}\Big\},\ (\tau_1,\tau_2)\in\mathcal{T}_2.$$

We omitted the integral term in our analysis because $L \equiv 0$ in problem (2).

Remark I.2. Remember the following assumption is in effect for this problem.

$$f(x^*) < K - p \tag{6}$$

and this condition implies that

$$p < K \tag{7}$$

since $f(x^*) \geq 0$.

1. *A recursive solution via dynamic programming*

Consider the following two optimal stopping problems:

$$\hat{v}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\{e^{-r\tau}L_2(X(\tau))\} \tag{8}$$

$$\hat{u}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(L_1(X(\tau)) + \hat{v}(X(\tau)))\} \tag{9}$$

Define two stopping sets $S_{\hat{v}}$ and $S_{\hat{u}}$ by

$$S_{\hat{v}} = \{x\in\mathbb{R} : \hat{v}(x) = L_2(x)\},\quad S_{\hat{u}} = \{x\in\mathbb{R} : \hat{u}(x) = L_1(x) + \hat{v}(x)\} \tag{10}$$

and two stopping times $D_{S_{\hat{v}}}$ and $D_{S_{\hat{u}}}$ by

$$D_{S_{\hat{v}}} = \inf\{t\geq 0 : X(t)\in S_{\hat{v}}\},\quad D_{S_{\hat{u}}} = \inf\{t\geq 0 : X(t)\in S_{\hat{u}}\}. \tag{11}$$

In equation (11) we use the convention $\inf\emptyset = \infty$. Theorem I.5 below provides an analytical characterisation of the solutions to the optimal stopping problems (8)–(9), and the connection between them and the starting–stopping problem. In order to do so, we now introduce the concept of an r-excessive function.

Definition 1. A measurable function $\Phi : \mathbb{R} \to [0,\infty]$ is said to be r-excessive (relative to X) if:

1. Φ is r-superaveraging:

$$e^{-rt}\mathbb{E}_x\{\Phi(X(t))\} \leq \Phi(x),\quad \forall x\in\mathbb{R} \text{ and } t\geq 0. \tag{12}$$

2. $\lim_{t\downarrow 0} e^{-rt}\mathbb{E}_x\{\Phi(X(t))\} = \Phi(x),\quad \forall x\in\mathbb{R}.$

The following well known properties of r-excessive functions can be found in [5] for instance.

Proposition I.3. *Suppose $\Phi\colon \mathbb{R}\to[0,\infty]$ is r-excessive (relative to X):*

- *Almost surely, the mapping $t\mapsto\Phi(X(t))$ is right-continuous on $[0,\infty)$ and has left-hand limits on $(0,\infty]$;*
- *If $\Phi(X(t))$ is integrable for each $t\geq 0$, then $(e^{-rt}\Phi(X(t)))_{t\geq 0}$ is a right-continuous supermartingale. Furthermore, for all $x\in\mathbb{R}$:*

$$\mathbb{E}_x\big\{e^{-rT}\Phi(X(T))\big\} \leq \mathbb{E}_x\big\{e^{-rS}\Phi(X(S))\big\},$$

for all stopping times S and T satisfying $S \leq T$ almost surely.

Remark I.4. In this paper, all referenced r-excessive functions are non-negative. In the special case of Brownian motion on $\mathbb{R}$, the r-excessive functions are actually *continuous* – see [1, p. 32] for instance.

Theorem I.5. *Suppose L_1 and L_2 are bounded and continuous functions. Then we have,*

i) The value function $\hat{v}$ is continuous, bounded and the smallest r-excessive majorant of L_2. Furthermore, $D_{S_{\hat{v}}}$ is an optimal stopping time in equation (8)*:*

$$\hat{v}(x) = \mathbb{E}_x\{e^{-rD_{S_{\hat{v}}}} L_2(X(D_{S_{\hat{v}}}))\} \quad \forall x \in \mathbb{R}.$$

ii) The value function $\hat{u}$ is continuous, bounded, and the smallest r-excessive majorant of $L_1 + \hat{v}$. Furthermore, $D_{S_{\hat{u}}}$ is an optimal stopping time in equation (9)*:*

$$\hat{u}(x) = \mathbb{E}_x\{e^{-rD_{S_{\hat{u}}}} (L_1(X(D_{S_{\hat{u}}})) + \hat{v}(X(D_{S_{\hat{u}}})))\} \quad \forall x \in \mathbb{R}.$$

iii) Let u be the value function of the starting–stopping problem (5)*. Then $\hat{u}$ in* (9) *satisfies $\hat{u} \geq u$, and if we define $(\hat{\tau}_1, \hat{\tau}_2) \in \mathcal{T}_2$ by:*

$$\hat{\tau}_1 = D_{S_{\hat{u}}}, \quad \hat{\tau}_2 = \hat{\tau}_1 + D_{S_{\hat{v}}} \circ \theta_{\hat{\tau}_1} \tag{13}$$

then $\hat{u}(x) = J_x(\hat{\tau}_1, \hat{\tau}_2) = u(x)$ for all $x \in \mathbb{R}$.

Proof. Proof of *i*).

We will apply results due to Dayanik and Karatzas [6]. Since L_2 is bounded we have

$$\limsup_{x\to\infty} \frac{L_2^+(x)}{\psi_r(x)} = \limsup_{x\to\infty} \frac{L_2^+(x)}{e^{ax}} = 0 \tag{14}$$

and

$$\limsup_{x\to-\infty} \frac{L_2^+(x)}{\phi_r(x)} = \limsup_{x\to-\infty} \frac{L_2^+(x)}{e^{-ax}} = 0. \tag{15}$$

Since L_2 is continuous and both equations (14) and (15) hold, we can use Propositions 5.11 and 5.13 of [6] to assert that $\hat{v}$ is the smallest r-excessive majorant of L_2, $\hat{v}$ is continuous, and $D_{S_{\hat{v}}}$ is an optimal stopping time in equation (8). Note that as L_2 is bounded, the constant function $Q(x) = \sup_z |L_2(z)| < \infty$ is an r-excessive majorant of L_2 and $\hat{v} \leq Q$.

Proof of *ii*).

Since $L_1 + \hat{v}$ is continuous and bounded, using the same arguments as before we can assert that $\hat{u}$ is continuous, bounded, and the smallest r-excessive majorant of $L_1 + \hat{v}$. Furthermore, $D_{S_{\hat{u}}}$ is an optimal stopping time in equation (9).

Proof of *iii*).

The proof proceeds in the same way as Theorem 3.1 in [4] and will just be sketched. We have already established the following:

- $\hat{v}$ is the smallest r-excessive majorant of L_2 and is also continuous and bounded;
- $\hat{u}$ is the smallest r-excessive majorant of $L_1 + \hat{v}$ and is also continuous and bounded.

Using this characterisation and Proposition I.3, for any pair of stopping times $(\tau_1, \tau_2) \in \mathcal{T}_2$ we have:

$$\begin{aligned}
\hat{u}(x) \geq \mathbb{E}_x\{e^{-r\tau_1}\hat{u}(X(\tau_1))\} &\geq \mathbb{E}_x\{e^{-r\tau_1} L_1(X(\tau_1)) + e^{-r\tau_1}\hat{v}(X(\tau_1))\} \\
&\geq \mathbb{E}_x\Big\{L_1(X(\tau_1))e^{-r\tau_1} + e^{-r\tau_2}\hat{v}(X(\tau_2))\Big\} \\
&\geq \mathbb{E}_x\Big\{L_1(X(\tau_1))e^{-r\tau_1} + e^{-r\tau_2} L_2(X(\tau_2))\Big\} = J_x(\tau_1, \tau_2).
\end{aligned}$$

This shows $\hat{u}(x) \geq \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} J_x(\tau_1,\tau_2)$. Let $(\hat{\tau}_1,\hat{\tau}_2) \in \mathcal{T}_2$ be given by Equation (13). Since $\hat{u}$ and $\hat{v}$ are the value functions of the optimal stopping problems (9) and (8) respectively, and the stopping times $\hat{\tau}_1$ and $\hat{\tau}_2$ are optimal in these respective problems, we can use the strong Markov property of X to show:

$$\begin{aligned}
\hat{u}(x) &= \mathbb{E}_x\Big\{e^{-r\hat{\tau}_1}(L_1(X(\hat{\tau}_1)) + \hat{v}(X(\hat{\tau}_1)))\Big\} \\
&= \mathbb{E}_x\Big\{e^{-r\hat{\tau}_1}L_1(X(\hat{\tau}_1)) + \mathbb{E}_x\Big\{e^{-r\hat{\tau}_2}L_2(X(\hat{\tau}_2)) \;\Big|\; \mathcal{F}_{\hat{\tau}_1}\Big\}\Big\} \\
&= \mathbb{E}_x\Big\{e^{-r\hat{\tau}_1}L_1(X(\hat{\tau}_1)) + e^{-r\hat{\tau}_2}L_2(X(\hat{\tau}_2))\Big\} = J_x(\hat{\tau}_1,\hat{\tau}_2)
\end{aligned}$$

which completes the proof. □

Let us now introduce the two optimal stopping problems associated with (2):

$$V_{II}(x) = \sup_{\tau_2\in\mathcal{T}} \mathbb{E}_x\{e^{-r\tau_2}(p + h_{III}(X(\tau_2)))\}, \quad x \in \mathbb{R} \tag{16}$$

$$\hat{V}_I(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(f(X(\tau)) + V_{II}(X(\tau)))\}, \quad x \in \mathbb{R} \tag{17}$$

From Theorem I.5 above we derive the following the result.

Corollary I.6. *The value functions V_{II} and $\hat{V}_I$ in* (16) *and* (17) *respectively are continuous, bounded and r-excessive. Furthermore, $\hat{V}_I$ is the solution to the optimal starting–stopping problem* (2)*:*

$$\hat{V}_I(x) = \sup_{(\tau_1,\tau_2)\in\mathcal{T}_2} \mathbb{E}_x\{e^{-r\tau_1}(f(X(\tau_1))) + e^{-r\tau_2}(p + h_{III}(X(\tau_2)))\} =: V_I(x).$$

II. EXPLICIT SOLUTIONS FOR THE SINGLE CONTRACT

This section provides the proofs of Proposition 3.1 and Theorem 3.1 from the main article. Both results make use of the following analytic characterisation of the value function to an optimal stopping problem given by [6].

Proposition II.1 (Proposition 5.12, [6])**.** *Let $F\colon \mathbb{R} \to (0,\infty)$ be defined by*

$$F(x) := \frac{\psi_r(x)}{\phi_r(x)} = e^{2ax}, \tag{18}$$

and $F^{-1}(\cdot) = \frac{\ln(\cdot)}{2a}$ denote its inverse. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a given continuous and bounded function, and $W : [0,\infty) \to [0,\infty)$ be the smallest non-negative concave majorant of

$$H(y) := \begin{cases} \frac{h(F^{-1}(y))}{\phi_r(F^{-1}(y))}, & y > 0 \\ 0, & y = 0. \end{cases}$$

Then the value function V of the optimal stopping problem

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\{e^{-r\tau}h(X(\tau))\}$$

satisfies $V(x) = \phi_r(x)W(F(x))$, and an optimal stopping time τ^* is given by

$$\tau^* = D_{S_V} := \inf\{t \geq 0 : X(t) \in S_V\}$$

where

$$S_V := \{x \in \mathbb{R} : V(x) = h(x)\} = \{x \in \mathbb{R} : \phi_r(x)W(F(x)) = h(x)\}.$$

A. Analysis of the decision to enter the contract

Theorem 3.1. *In regime II, the value function V_{II} (cf. (16)) is given explicitly by*

$$V_{II}(x) = \begin{cases} p - Ke^{a(x-x^*)}, & -\infty < x \leq \frac{\ln(p/2K)}{a} + x^* \\ e^{a(x^*-x)}\frac{p^2}{4K}, & \frac{\ln(p/2K)}{a} + x^* < x. \end{cases} \tag{19}$$

Moreover, the set S_{II} of values for the instantaneous imbalance at which it is optimal to immediately enter the contract is defined by:

$$S_{II} = (-\infty, \tfrac{\ln(p/2K)}{a} + x^*] \tag{20}$$

Proof. The result is a straightforward application of Proposition II.1. Using the notation $y^* = F(x^*) = e^{2ax^*}$ (cf. 18) we get:

$$H_1(y) = \frac{p + h_{III}(\frac{\ln(y)}{2a})}{y^{-\frac{1}{2}}} = \begin{cases} \hat{g}_1(y), & y^* < y \\ g_1(y), & 0 < y \leq y^* \\ 0, & y = 0 \end{cases} \tag{21}$$

where g_1 and $\hat{g}_1$ are real-valued functions on $(0, \infty)$ defined by

$$g_1(y) = \sqrt{y}p - y\frac{K}{\sqrt{y^*}}, \quad \hat{g}_1(y) = \sqrt{y}p - \sqrt{y}K. \tag{22}$$

We now determine the smallest non-negative concave majorant of H_1. Differentiating g_1 and $\hat{g}_1$ we get:

$$g_1'(y) = \frac{1}{2}\frac{p}{\sqrt{y}} - \frac{K}{\sqrt{y^*}}, \quad g_1''(y) = -\frac{1}{4}py^{-\frac{3}{2}} \tag{23}$$

$$\hat{g}_1'(y) = \frac{1}{2}\left(\frac{p}{\sqrt{y}} - \frac{K}{\sqrt{y}}\right), \quad \hat{g}_1''(y) = -\frac{1}{4}py^{-\frac{3}{2}} + \frac{1}{4}Ky^{-\frac{3}{2}} \tag{24}$$

Note that the sustainability condition (6) and non-negativity of f imply $0 < p < K$. From equations (22) and (24), we see that $\hat{g}_1(y)$ is a negative, monotonically decreasing and convex function on $(0, \infty)$. On the other hand, by equation (23) we see that g_1 is concave on $(0, \infty)$. The unique roots of g_1 and g_1' are respectively:

$$Y_a = \left(\frac{p\sqrt{y^*}}{K}\right)^2, \quad Y_{II} = \left(\frac{p\sqrt{y^*}}{2K}\right)^2 \tag{25}$$

and these points satisfy $Y_{II} < Y_a < y^*$ since $p < K$. Recalling equation (22), we see that g_1 is:

1. concave, increasing and positive on $(0, Y_{II}]$
2. concave, decreasing and positive on (Y_{II}, Y_a)
3. concave, decreasing and negative on (Y_a, ∞).

From this it follows that H_1 (cf. (21)) is:

1. concave, increasing and positive on $(0, Y_{II}]$
2. concave, decreasing and positive on (Y_{II}, Y_a)
3. concave, decreasing and negative on (Y_a, y^*)
4. convex, decreasing and negative on (y^*, ∞)

We conclude that the smallest non-negative concave majorant of H_1 is:

$$W(y) = \begin{cases} g_1(y), & 0 \leq y \leq Y_{II} \\ g_1(Y_{II}) = \frac{\sqrt{y^*}p^2}{4K}, & Y_{II} < y. \end{cases}$$

Finally, by applying Proposition II.1 we get:

$$V_{II}(x) = \begin{cases} e^{-ax} g_1(e^{2ax}) = p - K e^{a(x-x^*)}, & -\infty < x \le \frac{\ln(p/2K)}{a} + x^* \\ e^{-ax} \frac{e^{ax^*} p^2}{4K} = e^{a(x^*-x)} \frac{p^2}{4K}, & \frac{\ln(p/2K)}{a} + x^* < x. \end{cases}$$

The stopping region $S_{II} := \{V_{II} = p + h_{III}\}$ is then given explicitly by

$$S_{II} = (-\infty, \tfrac{\ln(p/2K)}{a} + x^*].$$

□

Remark II.2. This result is unchanged if we drop the sustainability condition (6) and keep only (7). However, if condition (7) is not satisfied (so $p > K$), the solution changes entirely. Indeed, let W be the smallest non-negative concave majorant of H_1 and define the (closed) stopping set $\tilde{\mathcal{S}}_W := \{y \in (0,\infty) \colon W(y) = H_1(y)\}$. By definition of H_1 and as g_1 is concave and non-negative on $(0, Y_a \wedge y^*]$, one can show that $\tilde{\mathcal{S}}_W \cap (0, y^*] \neq \emptyset$ and furthermore $\tilde{\mathcal{S}}_W \cap (0, y^*] = (0, \hat{y}]$ for some $0 < \hat{y} \le y^*$. Since $p > K$ is assumed, we have $\hat{g}_1'(y^*) > g_1'(y^*)$ and the transformed obstacle H_1 is not concave in some open interval containing the point y^*. The function W therefore strictly dominates H_1 on this interval. If $\tilde{\mathcal{S}}_W \cap (y^*, \infty) = \emptyset$, then W would be linear on $[y^*, \infty)$ (see [6]). Using the definition of H_1, W is the smallest non-negative concave majorant of H_1, and $\hat{g}_1$ is a positive, concave and increasing function on $(0,\infty)$ that satisfies $\lim_{y\to\infty} \hat{g}_1(y) = \infty$, one can show that there must be a point $y \ge y^*$ such that W is tangent to $\hat{g}_1$ and $W(y) = \hat{g}_1(y) = H_1(y)$, which is a contradiction. Therefore, $\tilde{\mathcal{S}}_W \cap (y^*, \infty) \neq \emptyset$ and using the concavity of $\hat{g}_1$ one can show that $\tilde{\mathcal{S}}_W \cap (y^*, \infty) = [\check{y}, \infty)$ for some $\check{y} > y^*$. We have therefore shown $\tilde{\mathcal{S}}_W = (0, \hat{y}] \cup [\check{y}, \infty)$, and the stopping set is disconnected when $p > K$.

B. Analysis of the decision to sell electricity

Recall that $y^* = F(x^*) = e^{2ax^*}$. Let X_{II} be defined by,

$$X_{II} = F^{-1}(Y_{II}) = \tfrac{\ln(p/2K)}{a} + x^*. \tag{26}$$

Let X_{min} and X_{max} denote the lower and upper caps for the market price in (3),

$$X_{min} = \frac{-c}{b}, \quad X_{max} = \frac{M-c}{b}, \tag{27}$$

and set $Y_{min} = F(X_{min}) = e^{\frac{-2ac}{b}}$, $Y_{max} = F(X_{max}) = e^{\frac{2a}{b}(M-c)}$. We will now analyse (17) using Proposition II.1, presenting the solution for the following case only.

Case 2: $X_{max} \le X_{II} \le X_{min}$

1. *Auxiliary functions in the transformed coordinates*

In order to assist in the analysis, we introduce some auxiliary functions defined on $(0,\infty)$. These functions are defined by:

$$G(y) = c\sqrt{y} + b\sqrt{y}\frac{\ln(y)}{2a} + \sqrt{y}p - y\frac{K}{\sqrt{y^*}}, \quad \hat{G}(y) = c\sqrt{y} + b\sqrt{y}\frac{\ln(y)}{2a} + \sqrt{y^*}\frac{p^2}{4K}$$
$$G_1(y) = M\sqrt{y} + \sqrt{y}p - y\frac{K}{\sqrt{y^*}}, \quad \hat{G}_1(y) = \sqrt{y^*}\frac{p^2}{4K}$$

Calculating the first and second derivatives of these functions gives,

$$G'(y) = \frac{1}{2\sqrt{y}}\Big(c + p + \frac{b}{a} + \frac{b\ln(y)}{2a}\Big) - \frac{K}{\sqrt{y^*}}, \quad G''(y) = -\frac{1}{4}y^{-\frac{3}{2}}(c + p + \frac{b\ln(y)}{2a})$$

$$\hat{G}'(y) = \frac{1}{2\sqrt{y}}(c + \frac{b}{a} + \frac{b\ln(y)}{2a}), \quad \hat{G}''(y) = -\frac{1}{4}y^{-\frac{3}{2}}(c + \frac{b\ln(y)}{2a})$$

$$G_1'(y) = \frac{1}{2\sqrt{y}}\Big(M + p\Big) - \frac{K}{\sqrt{y^*}}, \quad G_1''(y) = -\frac{1}{4}y^{-\frac{3}{2}}(M + p)$$

$$\hat{G}_1'(y) = 0, \quad \hat{G}_1''(y) = 0$$

1. The function G'' is continuous on $(0, \infty)$, has a unique root point at $Y_1 = e^{\frac{-2a}{b}(c+p)}$, is negative on $(0, Y_1)$, and positive on (Y_1, ∞). Since G' is continuous, monotone and decreasing on $(0, Y_1)$, $\lim_{y\downarrow 0} G'(y) = \infty$ and $G'(Y_1) < 0$, G' has a unique root $Y_\Gamma \in (0, Y_1)$. Furthermore, since G' is continuous, monotone and increasing on (Y_1, ∞) with $\lim_{y\uparrow\infty} G'(y) = -\frac{K}{\sqrt{y^*}} < 0$, we see that G' is negative on (Y_Γ, ∞) with its minimum at Y_1.
2. The function G is concave and increasing on $(0, Y_\Gamma)$; concave and decreasing on (Y_Γ, Y_1); convex and decreasing on (Y_1, ∞).
3. The function $\hat{G}'$ is continuous on $(0, \infty)$ and has a unique root point at $Y_2 = e^{\frac{-2ac}{b} - 2}$. Furthermore, $\hat{G}'$ is positive on $(0, Y_2)$ and negative on (Y_2, ∞). Its derivative, $\hat{G}''$, has a unique root at $Y_{min} < Y_1$, is negative on $(0, Y_{min})$ and positive on (Y_{min}, ∞). Noting that $Y_2 < Y_{min}$.
4. The function $\hat{G}$ is concave and increasing on $(0, Y_2]$; concave and decreasing on (Y_2, Y_{min}); convex and decreasing on (Y_{min}, ∞). Furthermore, $\hat{G}(Y_2) = \frac{-2be^{-\frac{ac}{b}-1}}{2a} + \sqrt{y^*}\frac{p^2}{4K} > \hat{G}_1(y)$ for all $y \in (0, \infty)$.
5. The function G_1 is concave on $(0, \infty)$. Its derivative, G_1', is continuous and monotone on $(0, \infty)$. Furthermore, since $\lim_{y\downarrow 0} G_1'(y) = \infty$ and $\lim_{y\uparrow\infty} G_1'(y) = -\frac{K}{\sqrt{y^*}} < 0$, G_1' has a unique root $Y_5 \in (0, \infty)$ given by $Y_5 = \Big(\frac{\sqrt{y^*}(M+p)}{2K}\Big)^2$. It then follows that G_1 is increasing on $(0, Y_5]$ and decreasing on (Y_5, ∞).

At the point Y_{max}, we have

$$\begin{aligned} G_1(Y_{max}) &= G(Y_{max}) = \sqrt{Y_{max}}(M + p - K\frac{\sqrt{Y_{max}}}{\sqrt{y^*}}), \\ G_1'(Y_{max}) &= \frac{1}{2\sqrt{Y_{max}}}\Big(M + p\Big) - \frac{K}{\sqrt{y^*}}, \quad G'(Y_{max}) = G_1'(Y_{max}) + \frac{b}{2\sqrt{Y_{max}}}. \end{aligned}$$

Therefore, there is continuous fit between G_1 and G at Y_{max}, but smooth fit fails since $G'(Y_{max}) < G_1'(Y_{max})$.

At the point Y_{II} we have,

$$\begin{aligned} G(Y_{II}) &= \hat{G}(Y_{II}) = c\sqrt{Y_{II}} + b\sqrt{Y_{II}}\frac{\ln(Y_{II})}{2a} + \sqrt{y^*}\frac{p^2}{4K}, \\ G'(Y_{II}) &= \hat{G}'(Y_{II}) = \frac{1}{2}Y_{II}^{-\frac{1}{2}}(c + \frac{b}{a} + \frac{b\ln(Y_{II})}{2a}). \end{aligned}$$

which shows there is continuous and smooth fit between G and $\hat{G}$ at Y_{II}.

At the point Y_{min}, $\hat{G}$ and $\hat{G}_1$ satisfy

$$\hat{G}(Y_{min}) = \hat{G}_1(Y_{min}) = \sqrt{y^*}\frac{p^2}{4K}, \quad \hat{G}'(Y_{min}) = e^{\frac{ac}{b}}\frac{b}{2a}, \quad \hat{G}_1'(Y_{min}) = 0$$

which shows there is continuous fit at Y_{min} but no smooth fit since $\hat{G}'(Y_{min}) < \hat{G}_1'(Y_{min})$.

2. *Solution in regime I in the transformed coordinates*

With these results we can solve the optimal stopping problem (17) for **Case 2**.

For **Case 2** we have $0 < Y_{max} \leq Y_{II} \leq Y_{min}$, and the transformed payoff $H_2 \colon [0, \infty) \to \mathbb{R}$ in this case is defined by:

$$H_2(y) = \frac{\{f + V_{II}\}(F^{-1}(y))}{y^{-\frac{1}{2}}} = \begin{cases} G_1(y), & 0 < y < Y_{max} \\ G(y), & Y_{max} \leq y \leq Y_{II} \\ \hat{G}(y), & Y_{II} \leq y \leq Y_{min} \\ \hat{G}_1(y), & Y_{min} < y \\ 0, & y = 0 \end{cases} \tag{28}$$

We introduce one additional point $X_\Gamma \in \mathbb{R}$ via the transformation $X_\Gamma = F^{-1}(Y_\Gamma)$, where Y_Γ was identified previously as the unique root of G' in $(0, Y_1)$. Since the function F^{-1} is monotone and strictly increasing, can assert that X_Γ is the unique solution in $(-\infty, -\frac{c+p}{b}]$ to $\Gamma(x) = 0$ with,

$$\Gamma(x) = G'(F(x)) = \frac{1}{2} e^{-ax} \left(c + p + \frac{b}{a} + bx \right) - K e^{-ax^*}.$$

Theorem 3.2. *Let S_I denote the set of values for the instantaneous imbalance at which it is optimal to immediately sell the stored electricity. The value function in regime I, V_I, and corresponding set S_I are given explicitly in* **Case 2** *by:*

- **Case 2.1** : *If $X_{max} < X_\Gamma < X_{II}$, then*

$$V_I(x) = \begin{cases} M + p - K e^{a(x - x^*)}, & -\infty < x \leq X_{max} \\ c + bx + p - K e^{a(x - x^*)}, & X_{max} < x \leq X_\Gamma \\ e^{-a(x - X_\Gamma)} (c + bX_\Gamma + p - K e^{a(X_\Gamma - x^*)}), & X_\Gamma < x \end{cases} \tag{29}$$

  and $S_I = (-\infty, X_\Gamma]$.

- **Case 2.2** : *If $X_\Gamma \leq X_{max}$, then*

$$V_I(x) = \begin{cases} M + p - K e^{a(x - x^*)}, & -\infty < x \leq X_{max} \\ e^{a(X_{max} - x)} (M + p - K e^{a(X_{max} - x^*)}), & X_{max} < x \end{cases} \tag{30}$$

  and $S_I = (-\infty, X_{max}]$.

- **Case 2.3** : *If $X_{II} \leq X_\Gamma$, then*

$$V_I(x) = \begin{cases} M + p - K e^{a(x - x^*)}, & -\infty < x < X_{max} \\ c + bx + p - K e^{a(x - x^*)}, & X_{max} \leq x \leq X_{II} \\ c + bx + e^{a(x^* - x)} \frac{p^2}{4K}, & X_{II} < x \leq X_{min} - \frac{1}{a} \\ -\frac{b}{a} e^{a(X_{min} - x) - 1} + e^{a(x^* - x)} \frac{p^2}{4K}, & X_{min} - \frac{1}{a} < x \end{cases} \tag{31}$$

  and $S_I = (-\infty, X_{min} - \frac{1}{a}]$.

Proof. Following Proposition II.1, we will characterise the smallest non-negative majorant W of H_2 defined in (28). From the properties deduced above, we know that:

- the function G_1 is concave everywhere and increasing on $(0, Y_5]$, with $Y_5 > Y_{max}$.
- the first derivatives of G, G_1 and $\hat{G}$ satisfy $G'(Y_{max}) < G_1'(Y_{max})$ and $G'(Y_{II}) > \hat{G}'(Y_{II})$.
- the function $\hat{G}$ is concave everywhere, increasing on $(0, Y_2]$ and decreasing on (Y_2, ∞), with $0 < Y_2 < Y_{min} < Y_1$. Furthermore, $\hat{G}(Y_2) > \hat{G}_1(y)$ for all $y \in (0, \infty)$.

Recalling equation (28) for H_2, we conclude that the (transformed) stopping region is $\{W = H_2\} = (0, A]$ where $A \leq Y_2$. We now distinguish different subcases in **Case 2**.

Case 2.1 : $Y_{max} < Y_\Gamma < Y_{II}$.

The stopping region is $(0, Y_\Gamma]$, and on $Y_\Gamma \le y$ the function W is constant with value $G(Y_\Gamma)$:

$$W(y) = \begin{cases} \sqrt{y}(M + p - K\frac{\sqrt{y}}{\sqrt{y^*}}), & 0 < y \le Y_{max} \\ \sqrt{y}(c + b\frac{\ln(y)}{2a} + p - K\frac{\sqrt{y}}{\sqrt{y^*}}), & Y_{max} < y \le Y_\Gamma \\ \sqrt{Y_\Gamma}(c + b\frac{\ln(Y_\Gamma)}{2a} + p - K\frac{\sqrt{Y_\Gamma}}{\sqrt{y^*}}), & Y_\Gamma < y \end{cases} \tag{32}$$

Finally, by applying Proposition II.1 we get:

$$V_I(x) = \begin{cases} M + p - Ke^{a(x-x^*)}, & -\infty < x \le X_{max} \\ c + bx + p - Ke^{a(x-x^*)}, & X_{max} < x \le X_\Gamma \\ e^{-a(x-X_\Gamma)}(c + bX_\Gamma + p - Ke^{a(X_\Gamma - x^*)}), & X_\Gamma < x \end{cases} \tag{33}$$

Case 2.2 : $Y_\Gamma \le Y_{max}$.

The stopping region is $(0, Y_{max}]$, and on $Y_{max} \le y$ the function W is constant with value $G(Y_{max})$:

$$W(y) = \begin{cases} \sqrt{y}(M + p - K\frac{\sqrt{y}}{\sqrt{y^*}}), & 0 < y \le Y_{max} \\ \sqrt{Y_{max}}(M + p - K\frac{\sqrt{Y_{max}}}{\sqrt{y^*}}), & Y_{max} < y \end{cases} \tag{34}$$

Finally, by applying Proposition II.1 we get:

$$V_I(x) = \begin{cases} M + p - Ke^{a(x-x^*)}, & -\infty < x \le X_{max} \\ e^{-ax}e^{\frac{a}{b}(M-c)}(M + p - Ke^{\frac{a}{b}(M-c) - ax^*}), & X_{max} < x \end{cases} \tag{35}$$

Case 2.3 : $Y_{II} \le Y_\Gamma$.
Note that $Y_{II} \le Y_\Gamma$ implies $\hat{G}'(Y_{II}) \ge 0$, since $G'(Y_{II}) = \hat{G}'(Y_{II})$ and G' is non-negative on $(0, Y_\Gamma]$. Furthermore, since $\hat{G}'$ is non-negative on $(0, Y_2]$, negative on (Y_2, ∞) and has a unique root at Y_2, we see that $\hat{G}'(Y_{II}) \ge 0$ implies $Y_{II} \le Y_2$. Therefore, $Y_{II} \le Y_\Gamma$ implies $Y_{II} \le Y_2$. Conversely, if $Y_{II} \le Y_2$ then $0 \le \hat{G}'(Y_{II}) = G'(Y_{II})$ and, since G' is non-negative on $(0, Y_\Gamma]$, negative on (Y_Γ, ∞) and has a unique root at Y_Γ, we get $Y_{II} \le Y_\Gamma$. Hence **Case 2.3** is equivalently expressed as **Case 2.3′**: $Y_{II} \le Y_2$. The stopping region in this case is given by $(0, Y_2]$, and for $y \ge Y_2$ the function W is constant with value $\hat{G}(Y_2)$:

$$W(y) = \begin{cases} \sqrt{y}(M + p - K\frac{\sqrt{y}}{\sqrt{y^*}}), & 0 < y < Y_{max} \\ \sqrt{y}(c + b\frac{\ln(y)}{2a} + p - K\frac{\sqrt{y}}{\sqrt{y^*}}), & Y_{max} \le y \le Y_{II} \\ \sqrt{y}(c + b\frac{\ln(y)}{2a} + \frac{\sqrt{y^*}}{\sqrt{y}}\frac{p^2}{4K}), & Y_{II} < y \le Y_2 \\ -\frac{b}{a}\sqrt{Y_2} + \sqrt{y^*}\frac{p^2}{4K}, & Y_2 < y \end{cases} \tag{36}$$

Finally, by applying Proposition II.1 we get:

$$V_I(x) = \begin{cases} M + p - Ke^{a(x-x^*)}, & -\infty < x < X_{max} \\ c + bx + p - Ke^{a(x-x^*)}, & X_{max} \le x \le X_{II} \\ c + bx + e^{a(x^*-x)}\frac{p^2}{4K}, & X_{II} < x \le X_{min} - \frac{1}{a} \\ -\frac{b}{a}e^{-\frac{ac}{b} - 1 - ax} + e^{a(x^*-x)}\frac{p^2}{4K}, & X_{min} - \frac{1}{a} < x \end{cases} \tag{37}$$

□

III. ANALYSIS OF THE LIFETIME VALUE

In this part we provide supporting results for Section 4 in the main article. Recall that the function V_I^n is defined inductively as the value function for the following starting–stopping problem:

$$V_I^n(x) = \sup_{(\tau_1, \tau_2) \in \mathcal{T}_2} \mathbb{E}_x\Big\{ f(X(\tau_1))e^{-r\tau_1} + (p + h_{III}^n(X(\tau_2)))e^{-r\tau_2} \Big\} \tag{38}$$

where h^n_{III} is defined by,

$$
\begin{aligned}
h^n_{III}(x) &:= \mathbb{E}_x\{e^{-rD_{[x^*,\infty)}}(V^{n-1}_I(X(D_{[x^*,\infty)})) - K)\} \\
&= \begin{cases} V^{n-1}_I(x) - K, & x^* < x \\ (V^{n-1}_I(x^*) - K)e^{a(x-x^*)}, & x \le x^* \end{cases}
\end{aligned} \tag{39}
$$

with $V^0_I \equiv 0$. For each $n \ge 1$, we define V^n_{II} as the value function of the following optimal stopping problem:

$$
V^n_{II}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(p + h^n_{III}(X(\tau)))\}. \tag{40}
$$

Using Theorem I.5, we can assert that the value function V^n_I for the optimal starting–stopping problem (38) satisfies

$$
V^n_I(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(f(X(\tau)) + V^n_{II}(X(\tau)))\}, \quad n \ge 1. \tag{41}
$$

Lemma III.1. *For all $n \ge 1$, V^n_I, V^n_{II} and h^n_{III} are continuous and bounded. Furthermore, the sequences of functions $\{V^n_I\}_{n\ge 0}$, $\{V^n_{II}\}_{n\ge 1}$ and $\{h^n_{III}\}_{n\ge 1}$ are nondecreasing.*

Proof. Step (i):

We first show that for all $n \ge 0$, the functions V^n_I, h^{n+1}_{III} and V^{n+1}_{II} are continuous and bounded. Note that $V^0_I \equiv 0$ is bounded and continuous. This implies the function h^1_{III} is continuous and bounded by definition, and Theorem I.5 above shows that both V^1_{II} and V^1_I are continuous and bounded. Therefore, for $n = 1$ we have V^{n-1}_I is continuous and bounded implies V^n_I, V^n_{II} and h^n_{III} are continuous and bounded. Assuming for a given $n \ge 1$ that V^{n-1}_I is continuous and bounded, we assert in the same way that V^n_I, V^n_{II} and h^n_{III} are continuous and bounded. We conclude by induction on $n \ge 1$.

Step (ii):

We now show that $\{V^n_I\}_{n\ge 0}$, $\{V^n_{II}\}_{n\ge 1}$ and $\{h^n_{III}\}_{n\ge 1}$ are nondecreasing sequences. Using Theorem I.5 we can show that the function V^1_I is non-negative, which shows $h^2_{III} \ge h^1_{III}$. Since V^2_I and V^1_I are solutions to their respective starting–stopping problems (38), $h^2_{III} \ge h^1_{III} \implies V^2_I \ge V^1_I$. Similarly, if $V^n_I \ge V^{n-1}_I$ for some $n \ge 1$, then $h^{n+1}_{III} \ge h^n_{III}$ and therefore $V^{n+1}_I \ge V^n_I$. The same reasoning shows $V^{n+1}_{II} \ge V^n_{II}$ since V^n_{II} is the solution to the optimal stopping problem (40). The proof is completed by induction on $n \ge 1$. □

Theorem 4.1. *The limiting functions $V^*_I = \lim_{n\to\infty} V^n_I$ and $V^*_{II} = \lim_{n\to\infty} V^n_{II}$ exist and are lower semicontinuous functions. Furthermore, V^*_I and V^*_{II} are bounded and:*

(i) V^*_I is the smallest r-excessive majorant of $f + V^*_{II}$.

(ii) V^*_{II} is the smallest r-excessive majorant of $p + h^*_{III}$, where

$$
h^*_{III}(x) = \begin{cases} V^*_I(x) - K, & x^* < x \\ (V^*_I(x^*) - K)e^{a(x-x^*)}, & x \le x^* \end{cases} \tag{42}
$$

(iii) V^*_I and V^*_{II} are both continuous and are the value functions of optimal stopping problems:

$$
V^*_{II}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(p + h^*_{III}(X(\tau)))\} \tag{43}
$$

$$
V^*_I(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(f(X(\tau)) + V^*_{II}(X(\tau)))\} \tag{44}
$$

Furthermore, V^*_I is the value function of an optimal starting-stopping problem:

$$
V^*_I(x) = \sup_{(\tau_1,\tau_2)\in \mathcal{T}_2} \mathbb{E}_x\{e^{-r\tau_1}(f(X(\tau_1))) + e^{-r\tau_2}(p + h^*_{III}(X(\tau_2)))\}. \tag{45}
$$

Proof. The first claim holds since V^*_I and V^*_{II} are the respective suprema of nondecreasing sequences of continuous and bounded functions. Proposition III.4 below confirms that V^*_I is bounded, from which we deduce h^*_{III} and V^*_{II} are bounded.

Proof of *i*). We know that for every fixed n, V_I^n is an r-excessive function. It is well known, for example [5, p. 81], that as the limiting function of an increasing sequence of r-excessive functions, V_I^* is also r-excessive. Since V_I^n is the smallest r-excessive majorant of $f + V_{II}^n$ for $n \geq 1$, we deduce $V_I^* \geq f + V_{II}^n$ for all $n \geq 1$. By taking the limit with respect to n, we get $V_I^* \geq f + V_{II}^*$, and V_I^* is an r-excessive majorant of $f + V_{II}^*$. In order to show that it is the smallest, let $\tilde{u}\colon \mathbb{R} \to [0, \infty]$ be any r-excessive function dominating $f + V_{II}^*$, so that

$$\tilde{u} \geq f + V_{II}^* \geq f + V_{II}^n, \quad \forall n \geq 1.$$

We already know that V_I^n is the smallest r-excessive majorant of $f + V_{II}^n$ for all $n \geq 1$. Therefore,

$$\tilde{u} \geq V_I^n, \quad \forall n \geq 1$$

which shows $\tilde{u} \geq \sup_n V_I^n = V_I^*$.

Proof of *ii*). The limiting function h_{III}^* in (42) exists and is bounded. The properties for V_{II}^* are then established in the same way as for V_I^*.

Proof of *iii*). Since V_{II}^* is the smallest r-excessive function majorising $p + h_{III}^*$, it is the value function of the optimal stopping problem:

$$V_{II}^*(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(p + h_{III}^*(X(\tau)))\}.$$

By Proposition 5.13 of [6], we see that V_{II}^* is continuous and bounded (also recall Remark I.4). We repeat these arguments to assert that V_I^* is also the value function of an optimal stopping problem,

$$V_I^*(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}(f(X(\tau)) + V_{II}^*(X(\tau)))\},$$

which is continuous and bounded. By applying Theorem I.5, we conclude that V_I^* is indeed the solution to an optimal starting–stopping problem of the form (45). □

A. Boundedness of the limiting value functions

We will now prove that the limiting function $V_I^* = \lim_{n\to\infty} V_I^n$ is bounded. Before proceeding, we require the following two lemmata.

Lemma III.2. *Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is continuous, bounded and decreasing after x^*: $h(x_1) \leq h(x_2)$ for all $x_1 > x_2 \geq x^*$. Consider the optimal stopping problem*

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\{e^{-r\tau}h(X(\tau))\}, \quad x \in \mathbb{R}$$

where $X = (X(t))_{t\geq 0}$ is a Brownian motion (starting from $x \in \mathbb{R}$) relative to $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_x)$. Then $V(x_1) \leq V(x_2)$ for all $x_1 > x_2 \geq x^*$.

Proof. Let $x_1 > x_2 \geq x^*$ be arbitrary. Let $S_V = \{x \in \mathbb{R}\colon V(x) = h(x)\}$ be the stopping region for the optimal stopping problem and $D_{S_V} = \inf\{t \geq 0\colon X(t) \in S_V\}$ denote its debut time. Note that V is continuous and bounded, and D_{S_V} is an optimal stopping time given our assumptions. If $S_V = \emptyset$ then $V \equiv 0$ and the claim holds trivially. Assume henceforth that $S_V \neq \emptyset$ and consider these two possibilities:

1. At least one of the two points x_1, x_2 belongs to S_V.
2. None of these points is in the stopping region.

Case 1.

If $x_1 \in S_V$, then $V(x_1) = h(x_1) \leq h(x_2) \leq V(x_2)$ by the assumption on h and since $V(x) \geq h(x)$ for all $x \in \mathbb{R}$. If $x_2 \in S_V$ then, since X has continuous trajectories, we must have $X(D_{S_V}) \geq x_2$ $\mathbb{P}_x$-almost surely for every $x > x_2$. By the decreasing property of h and as $x_2 \in S_V$, it follows that $h(X(D_{S_V})) \leq h(x_2) = V(x_2)$ $\mathbb{P}_x$-almost surely for every $x > x_2$, including x_1. Using the optimality of D_{S_V} together with $x_2 \in S_V$ and non-negativity of $V(\cdot)$, the claim is proved since

$$V(x_1) = \mathbb{E}_{x_1}\{e^{-rD_{S_V}}h(X(D_{S_V}))\} \leq \mathbb{E}_{x_1}\{e^{-rD_{S_V}}V(x_2)\} \leq V(x_2).$$

Case 2.

Suppose now that neither x_1 nor x_2 is in the stopping region. We will distinguish subcases depending on the form of the stopping region.

Case 2-i):

First assume that $S_V \cap (x_2, x_1) \neq \emptyset$. Let us choose an arbitrary $x_4 \in S_V \cap (x_2, x_1)$. We will show that $V(x_2) \geq V(x_4) \geq V(x_1)$. Since $x_4 \in S_V$ we have $V(x_4) = h(x_4)$ and, by the decreasing property of h, we also have $h(x_2) \geq h(x_4)$. Since $V(x) \geq h(x)$ for all $x \in \mathbb{R}$ we can conclude that $V(x_2) \geq V(x_4)$. Further, since X has continuous trajectories, we must have $X(D_{S_V}) \geq x_4$ $\mathbb{P}_x$-almost surely for every $x > x_4$. By the decreasing property of h and as $x_4 \in S_V$, it follows that $h(X(D_{S_V})) \leq h(x_4) = V(x_4)$ $\mathbb{P}_x$-almost surely for every $x > x_4$, including x_1. Arguing as we did in Case 1, the claim is proved since

$$V(x_1) = \mathbb{E}_{x_1}\{e^{-rD_{S_V}} h(X(D_{S_V}))\} \leq \mathbb{E}_{x_1}\{e^{-rD_{S_V}} V(x_4)\} \leq V(x_4) \leq V(x_2).$$

Case 2-ii):

Now, consider the subcase when $S_V \cap (x_2, x_1) = \emptyset$. Note that $S_V \cap (-\infty, x^*] \neq \emptyset$. Indeed, if $S_V \cap (-\infty, x^*] = \emptyset$ then we must have $S_V \subseteq [\ell, \infty)$ for some $\ell \in S_V \cap (x^*, \infty)$ since S_V is closed (as the level set of a continuous function) and non-empty. Using the optimality of D_{S_V} and $0 \leq V(\ell) = h(\ell)$, we get

$$h(x^*) < V(x^*) = \mathbb{E}_{x^*}[e^{-rD_{S_V}} h(X(D_{S_V}))] = \mathbb{E}_{x^*}[e^{-rD_{\{\ell\}}} h(\ell)] \leq h(\ell)$$

and this contradicts the assumption that h is decreasing on $[x^*, \infty)$. Therefore $S_V \cap (-\infty, x^*] \neq \emptyset$.

Suppose first that $S_V \subset (-\infty, x_2)$. Since S_V is closed and non-empty there must exist a point $b \in S_V$ such that $S_V \subseteq (-\infty, b]$. The value function therefore satisfies

$$\begin{aligned}
V(x_1) &= \mathbb{E}_{x_1}\{e^{-rD_{\{b\}}} h(b)\} = \mathbb{E}_{x_1}\{e^{-rD_{\{b\}}} V(b)\} = e^{a(b-x_1)} V(b) \\
V(x_2) &= \mathbb{E}_{x_2}\{e^{-rD_{\{b\}}} h(b)\} = \mathbb{E}_{x_2}\{e^{-rD_{\{b\}}} V(b)\} = e^{a(b-x_2)} V(b)
\end{aligned}$$

which shows $V(x_1) \leq V(x_2)$.

Finally, suppose that $S_V \subseteq (-\infty, \ell_1] \cup [\ell_2, \infty)$, where $\ell_1 < x_2 < x_1 < \ell_2$ and $\ell_1, \ell_2 \in S_V$ (remember $S_V \cap (-\infty, x^*] \neq \emptyset$). Using Lemma 4.3 of [6] we get the following: $\forall x \in (\ell_1, \ell_2)$,

$$\begin{aligned}
V(x) &= \mathbb{E}_x\{e^{-r(D_{\{\ell_1\}} \wedge D_{\{\ell_2\}})} h(X(D_{\{\ell_1\}} \wedge D_{\{\ell_2\}}))\} \\
&= h(\ell_1)\mathbb{E}_x\{e^{-rD_{\{\ell_1\}}} \mathbf{1}_{\{D_{\{\ell_1\}} < D_{\{\ell_2\}}\}}\} + h(\ell_2)\mathbb{E}_x\{e^{-rD_{\{\ell_2\}}} \mathbf{1}_{\{D_{\{\ell_2\}} < D_{\{\ell_1\}}\}}\} \\
&= h(\ell_1)u_1(x) + h(\ell_2)u_2(x) \qquad (46)
\end{aligned}$$

where

$$u_1(x) = \frac{e^{a(x-\ell_2)} - e^{a(\ell_2 - x)}}{e^{a(\ell_1-\ell_2)} - e^{a(\ell_2-\ell_1)}}, \quad u_2(x) = \frac{e^{a(\ell_1-x)} - e^{a(x-\ell_1)}}{e^{a(\ell_1-\ell_2)} - e^{a(\ell_2-\ell_1)}}. \qquad (47)$$

Our aim is to show that V is decreasing on (ℓ_1, ℓ_2), by showing that the derivative of V is non-positive on (ℓ_1, ℓ_2). Upon differentiating in (46), we get

$$V'(x) = a\frac{h(\ell_1)(e^{a(\ell_2-x)} + e^{a(x-\ell_2)}) - h(\ell_2)(e^{a(x-\ell_1)} + e^{a(\ell_1-x)})}{e^{a(\ell_1-\ell_2)} - e^{a(\ell_2-\ell_1)}}, \quad \forall x \in (\ell_1, \ell_2) \qquad (48)$$

which shows $x \mapsto V'(x)$ is continuous on (ℓ_1, ℓ_2), and the denominator in this expression is a negative constant. Note that for all $x \in (\ell_1, \ell_2)$, we have

$$\begin{aligned}
\Big(h(\ell_1)(e^{a(\ell_2-x)} + e^{a(x-\ell_2)}) - h(\ell_2)(e^{a(x-\ell_1)} + e^{a(\ell_1-x)})\Big)' = &-a\Big(h(\ell_1)(e^{a(\ell_2-x)} - e^{a(x-\ell_2)}) \\
&+ h(\ell_2)(e^{a(x-\ell_1)} - e^{a(\ell_1-x)})\Big) \\
&\leq 0,
\end{aligned}$$

where we used $h(\ell_1) = V(\ell_1)$, $h(\ell_2) = V(\ell_2)$ and $V(\cdot)$ is non-negative. Therefore, the numerator in (48), as a function of x, is decreasing on (ℓ_1, ℓ_2) and has a minimum on this interval at ℓ_2. Consequently, if the numerator in (48) is non-negative at ℓ_2, then we can conclude that it is non-negative on the whole interval (ℓ_1, ℓ_2). To this end, note that for all $\epsilon \in (0, \ell_2 - x^*)$, $0 \leq V(\ell_2) = h(\ell_2) \leq h(\ell_2 - \epsilon) \leq V(\ell_2 - \epsilon)$. Thus $V'(\ell_2^-) = \lim_{\epsilon \to 0} \frac{V(\ell_2) - V(\ell_2 - \epsilon)}{\epsilon} \leq 0$ and the numerator in (48) is not negative at $x = \ell_2$ (where we used its continuity). Therefore, the numerator in (48) is non-negative on all of (ℓ_1, ℓ_2), from which we conclude V' is non-positive and V is indeed decreasing on $[x^*, \infty)$ in this case. □

Lemma III.3. *Let $\xi \colon \mathbb{R} \to \mathbb{R}$ be continuous and bounded, decreasing on $[x^*, \infty)$ and satisfy $\xi(x^*) > K$. Consider the optimal stopping problem,*

$$h_1^\xi(x) = \sup_{\tau_2} \mathbb{E}_x\{e^{-r\tau_2}(p + h_2^\xi(X(\tau_2)))\} \tag{49}$$

where $h_2^\xi \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$h_2^\xi(x) = \mathbb{E}_x\{e^{-rD_{[x^*,\infty)}}(\xi(X(D_{[x^*,\infty)})) - K)\} = \begin{cases} \xi(x) - K, & x > x^* \\ (\xi(x^*) - K)e^{a(x - x^*)}, & x \leq x^* \end{cases}$$

Then $(-\infty, x^*] \subseteq D_{S_{h_1^\xi}} := \{h_1^\xi = p + h_2^\xi\}$.

Proof. Note that $p + h_2^\xi$ is a continuous and bounded function, so that we can apply Proposition II.1 above. Define $H \colon [0, \infty) \to \mathbb{R}$ by:

$$H(y) = \frac{p + h_2^\xi(\frac{\ln(y)}{2a})}{y^{-\frac{1}{2}}} = \begin{cases} \sqrt{y}(p + \frac{(\xi(x^*) - K)\sqrt{y}}{e^{ax^*}}), & 0 < y \leq y^* \\ \sqrt{y}(p + \xi(\frac{\ln(y)}{2a}) - K), & y^* < y \\ 0, & y = 0. \end{cases}$$

where $y^* = F(x^*) = e^{2ax^*}$ (cf. 18). Let $\hat{g}_2$ and $\hat{g}_1$ be two functions defined on $(0, \infty)$ by

$$\hat{g}_2(y) = \sqrt{y}(p + \tfrac{(\xi(x^*) - K)\sqrt{y}}{e^{ax^*}}), \quad \hat{g}_1(y) = \sqrt{y}(p + \xi(\tfrac{\ln(y)}{2a}) - K). \tag{50}$$

Here, $\hat{g}_2 = H$ on $[0, y^*]$ whereas $\hat{g}_1 = H$ on $[y^*, \infty)$. Differentiating $\hat{g}_2$ yields,

$$\hat{g}_2'(y) = \frac{1}{2}\frac{p}{\sqrt{y}} + \frac{\xi(x^*) - K}{e^{ax^*}}, \quad \hat{g}_2''(y) = -\frac{1}{4}py^{-\frac{3}{2}}. \tag{51}$$

Since $\xi(x^*) > K$, from equations (50) and (51) we see that $\hat{g}_2$ is a non-negative, concave and increasing function. Therefore the smallest non-negative concave majorant of $\hat{g}_2$ is itself. Furthermore, $\hat{g}_2$ majorises H since $\hat{g}_2(y) \geq \hat{g}_1(y) = H(y)$ on $[y^*, \infty)$:

$$\hat{g}_2(y) - \hat{g}_1(y) = \frac{y}{e^{ax^*}}(\xi(x^*) - K) - \sqrt{y}(\xi(\frac{\ln(y)}{2a}) - K) \geq \sqrt{y}((\frac{\sqrt{y}}{e^{ax^*}} - 1)(\xi(x^*) - K)) \geq 0$$

where we used $x \mapsto \xi(x)$ is decreasing on $[x^*, \infty)$ and $\sqrt{y} > \sqrt{y^*} = e^{ax^*}$ on (y^*, ∞). The smallest non-negative concave majorant of H, W, therefore satisfies $W \leq \hat{g}_2$. However, since we have $H = \hat{g}_2$ on $[0, y^*]$, the prior analysis shows $W = \hat{g}_2 = H$, $\forall y \in [0, y^*]$. Reverting to the original coordinate system shows $(-\infty, x^*] \subseteq D_{S_{h_1^\xi}}$. □

Proposition III.4. *The limiting function V_I^* is bounded.*

Proof. We distinguish two cases:

1. $V_I^n(x^*) \leq K$ for all $n \geq 1$
2. there exists an integer $n \geq 1$ such that $V_I^n(x^*) > K$

Case 1.

First, we will use Lemma III.2 and an induction argument to show that for all $n \geq 1$, $V_I^n(x^*) \geq V_I^n(x)$ for all $x \geq x^*$. This property is clearly true for $n = 0$. Let us assume it holds for $n = m \geq 1$. Then the payoff function $p + h_{III}^m$ is decreasing on $[x^*, \infty)$ and, by Lemma (III.2), the same holds for $V_{II}^{m+1}(x)$. Since $f(\cdot)$ is also decreasing on $\mathbb{R}$, we can use the characterisation of V_I^{m+1} in (41) and Lemma III.2 again to conclude that V_I^{m+1} is decreasing on $[x^*, \infty)$.

Since $V_I^n(x^*) \leq K$ for all $n \geq 0$, by definition we get $h_{III}^n(x) \leq 0$ on $(-\infty, x^*]$ for all $n \geq 1$. However, using the previous argument we have $h_{III}^n(x) = V_I^{n-1}(x) - K \leq V_I^n(x^*) - K \leq 0$ on (x^*, ∞). Hence $h_{III}^n(x) \leq 0$ for all $x \in \mathbb{R}$. Since $f(\cdot) \leq M$ and $p > 0$ is constant, we can then use the characterisation of V_I^n in equation (38) to conclude that for all $n \geq 1$, $V_I^n \leq M + p$, and therefore $V_I^* \leq M + p$.

Case 2.

For each $n \geq 1$, define the stopping sets S_{II}^n and S_I^n for the problems (40) and (41) respectively by

$$\begin{cases} S_{II}^n = \{x \in \mathbb{R} \colon V_{II}^n(x) = p + h_{III}^n(x)\} \\ S_I^n = \{x \in \mathbb{R} \colon V_I^n(x) = f(x) + V_{II}^n(x)\} \end{cases} \tag{52}$$

Next, define stopping times τ_1^n and τ_2^n by

$$\tau_1^n = D_{S_I^n}, \quad \tau_2^n = \tau_1^n + D_{S_{II}^n} \circ \theta_{\tau_1^n} \tag{53}$$

where $D_{S_{II}^n}$ and $D_{S_I^n}$ are the debut times of X into the sets S_{II}^n and S_I^n respectively. By Theorem I.5, for all $n \geq 1$ we have

$$V_I^n(x) = \mathbb{E}_x\Big\{e^{-r\tau_1^n} f(X(\tau_1^n)) + e^{-r\tau_2^n}(p + h_{III}^n(X(\tau_2^n)))\Big\} \tag{54}$$

Let us denote by n_0 the first $n \geq 1$ for which $V_I^n(x^*) > K$. Using the initial arguments in Case 1 above, we know that for all $n \geq n_0$ we have $h_{III}^{n+1}(x) \leq h_{III}^{n+1}(x^*)$ for all $x \geq x^*$. Further, by definition of $h_{III}^{n+1}(x)$ and using $V_I^n(x^*) > K$ for $n \geq n_0$, we see that x^* is actually a *global maximum* for $h_{III}^{n+1}(x)$ on $\mathbb{R}$. From this and (40), one can show:

$$\forall n \geq n_0 : \quad \begin{cases} V_{II}^{n+1}(x^*) = p + h_{III}^{n+1}(x^*) \\ V_{II}^{n+1}(x^*) \geq V_{II}^{n+1}(x) \ \ \forall x \in \mathbb{R} \end{cases}$$

The second line can be verified simply by noticing for all $n \geq n_0$, for any $x \in \mathbb{R}$ and every $\tau \in \mathcal{T}$,

$$\mathbb{E}_x\{e^{-r\tau}(p + h_{III}^{n+1}(X(\tau)))\} \leq \mathbb{E}_x\{e^{-r\tau}(p + h_{III}^{n+1}(x^*))\} = \mathbb{E}_x\{e^{-r\tau}(V_{II}^{n+1}(x^*))\} \leq V_{II}^{n+1}(x^*)$$

since V_{II}^{n+1} is non-negative. The claim is proved by taking supremum over all $\tau \in \mathcal{T}$ and using the arbitrariness of x. Finally, note that S_I^n is not empty for all $n \geq n_0$ since, if $S_I^{n+1} = \emptyset$ for some $n \geq n_0$, then the value function V_I^{n+1} would be zero everywhere and would contradict the assumption $V_I^{n+1}(x^*) > K$. Note that this also means that the stopping time $D_{S_I^{n+1}}$ is almost surely finite in this case (since $(X(t))_{t\geq 0}$ is a regular diffusion).

Step i)

We will show that the stopping region $S_I^{n+1} \cap (-\infty, x') \neq \emptyset$ for all $n \geq n_0$, where $x' < x^*$ is a fixed threshold. Using the sustainability condition (6) and $x \mapsto f(x)$ is continuous and monotonically decreasing, there exists an $x' < x^*$ such that $f(x') + p - K < 0$. Let us assume that there is an $n \geq n_0$ such that $S_I^{n+1} \cap (-\infty, x') = \emptyset$. Then this would imply $S_I^{n+1} \subseteq [x', \infty)$ and, from this and the previously mentioned properties of $x \mapsto f(x)$, we see that $f(X(\tau_1^{n+1})) \leq f(x')$ almost surely $\mathbb{P}_x$ for every x. Using the starting–stopping characterisation in (54) and h_{III}^{n+1} has a global maximum at x^* for all $n \geq n_0$, we have for all $x \geq x'$:

$$\begin{aligned} V_I^{n+1}(x) &= \mathbb{E}_x\Big\{e^{-r\tau_1^{n+1}} f(X(\tau_1^{n+1})) + e^{-r\tau_2^{n+1}}(p + h_{III}^{n+1}(X(\tau_2^{n+1})))\Big\} \\ &\leq \mathbb{E}_x\Big\{e^{-r\tau_1^{n+1}} f(x') + e^{-r\tau_2^{n+1}}(p + h_{III}^{n+1}(x^*)\Big\} \\ &= \mathbb{E}_x\Big\{e^{-r\tau_1^{n+1}} f(x') + e^{-r\tau_2^{n+1}}(p + (V_I^n(x^*) - K)\Big\} \\ &\leq f(x') + p + V_I^n(x^*) - K < V_I^n(x^*) \end{aligned} \tag{55}$$

where we also used $f(x') \geq 0$, $f(x') + p - K < 0$, $p > 0$ and $V_I^n(x^*) > K$ since $n \geq n_0$. In particular, this argument shows $V_I^{n+1}(x^*) < V_I^n(x^*)$ and contradicts that for every n we have $V_I^n(x) \leq V_I^{n+1}(x)$ for all x. Hence, we must have $S_I^{n+1} \cap (-\infty, x') \neq \emptyset$ for all $n \geq n_0$.

Step ii)

Let $m \geq 1$ be arbitrary. First, notice that $x^* \notin S_I^{m+n_0}$, since we would otherwise receive a contradiction:

$$V_I^{m+n_0}(x^*) = f(x^*) + V_{II}^{m+n_0}(x^*) = f(x^*) + p - K + V_I^{m-1+n_0}(x^*) < V_I^{m-1+n_0}(x^*).$$

We now show that there is at least one point $x \in S_I^{m+n_0}$ such that $V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x)$. Suppose contrarily that this were not true. Then $V_I^{m+n_0}(x^*) > V_I^{m+n_0}(x)$ for all $x \in S_I^{m+n_0}$ since $x^* \notin S_I^{m+n_0}$, which leads to the following contradiction:

$$\begin{aligned} V_I^{m+n_0}(x^*) &= \mathbb{E}_{x^*}\left\{ e^{-rD_{S_I^{m+n_0}}} V_I^{m+n_0}(X(D_{S_I^{m+n_0}})) \right\} \\ &< \mathbb{E}_{x^*}\left\{ e^{-rD_{S_I^{m+n_0}}} V_I^{m+n_0}(x^*) \right\} \leq V_I^{m+n_0}(x^*). \end{aligned}$$

Therefore for every $m \geq 1$ there exists a point $x \in S_I^{m+n_0}$ such that $V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x)$.

Step iii)

We now claim that for all $m \geq 1$ there must be at least one point $x_{m+n_0} \in S_I^{m+n_0} \cap (-\infty, x']$ such that $V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x_{m+n_0})$. Consider one of these two subcases:

1. $S_I^{m+n_0} \cap (x', \infty) = \emptyset$

2. $S_I^{m+n_0} \cap (x', \infty) \neq \emptyset$

Subcase 1.

The claim holds trivially in this case since we have already proved there is at least one point $x \in S_I^{m+n_0}$ such that $V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x)$ for $m \geq 1$.

Subcase 2.

Let $m \geq 1$ be arbitrary. We will show for all $x \in S_I^{m+n_0} \cap (x', \infty)$ that $V_I^{m+n_0}(x) < V_I^{m+n_0}(x^*)$. For all $x \in S_I^{m+n_0} \cap (x', \infty)$ we have $V_I^{m+n_0}(x) = f(x) + V_{II}^{m+n_0}(x)$. Our previous arguments have already shown that $V_{II}^{m+n_0}(x)$ has its global maximum at x^*, from which we deduce

$$\begin{aligned} V_I^{m+n_0}(x) \leq f(x) + V_{II}^{m+n_0}(x^*) &= f(x) + p + V_I^{m+n_0-1}(x^*) - K \\ &\leq f(x') + p + V_I^{m+n_0-1}(x^*) - K < V_I^{m+n_0-1}(x^*) \leq V_I^{m+n_0}(x^*) \end{aligned}$$

where we also used $f(x') + p - K < 0$, f is monotone decreasing and $\{V_I^n\}_{n \geq 0}$ is nondecreasing. However, we know that there is at least one point $x \in S_I^{m+n_0}$ such that $V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x)$. The previous reasoning shows that we must necessarily have $x \in S_I^{m+n_0} \cap (-\infty, x']$ in this case.

Step iv)

We have just shown for all $m \geq 1$ that there is at least one point $x_{m+n_0} \in S_I^{m+n_0} \cap (-\infty, x']$ such that $V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x_{m+n_0})$. Choosing one such point $x_{m+n_0} \in S_I^{m+n_0} \cap (-\infty, x']$ and using $x_{m+n_0} \in (-\infty, x^*] \subseteq S_{II^{m+n_0}}$ (cf. Lemma III.3), we receive:

$$\begin{aligned} V_I^{m+n_0}(x^*) \leq V_I^{m+n_0}(x_{m+n_0}) &= f(x_{m+n_0}) + V_{II}^{m+n_0}(x_{m+n_0}) \\ &= f(x_{m+n_0}) + p + h_{III}^{m+n_0}(x_{m+n_0}) \\ &\leq M + p + e^{a(x_{m+n_0} - x^*)}(V_I^{m-1+n_0}(x^*) - K) \\ &\leq M + p + e^{a(x' - x^*)}(V_I^{m-1+n_0}(x^*) - K) \end{aligned}$$

where we used $V_I^{m-1+n_0}(x^*) - K > 0$. Repeating these inequalities and using the properties of the geometric series, we can show for $m \geq 1$,

$$\begin{aligned} V_I^{m+n_0}(x^*) &\leq M + p + e^{a(x'-x^*)}(V_I^{m-1+n_0}(x^*) - K) \\ &\leq (M+p)\sum_{k=1}^{m} e^{a(m-k)(x'-x^*)} + e^{am(x'-x^*)}V_I^{n_0}(x^*) \\ &= (M+p)\frac{1-q^m}{1-q} + e^{am(x'-x^*)}V_I^{n_0}(x^*) \\ &\leq (M+p)\frac{1}{1-q} + V_I^{n_0}(x^*) =: T \end{aligned} \tag{56}$$

where $0 < q = e^{a(x'-x^*)} < 1$.

We have just shown in (56) that $V_I^n(x^*) \leq T < \infty$ for all $n \geq n_0$. We now claim that for all $n \geq n_0$, $V_I^n(x) \leq T + M + p$ for all $x \in \mathbb{R}$. First recall that for $n \geq n_0$, $V_{II}^{n+1}(x^*) = p + V_I^n(x^*) - K$ and $V_{II}^{n+1}(x) \leq V_{II}^{n+1}(x^*)$ for all $x \in \mathbb{R}$. Using (56), this means $V_{II}^{n+1} \leq T + p$. Since V_I^{n+1} is the smallest r-excessive majorant of $f + V_{II}^{n+1}$, f is bounded from above by M, and $\{V_I^n\}_{n\geq 0}$ is nondecreasing, we have $V_I^n \leq V_I^{n+1} \leq T + p + M$ for all $n \geq n_0$. Finally, taking the limit as $n \to \infty$ shows that $V_I^* \leq T + p + M$. □

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