Supplement to: Lamb's Problem on Random Mass Density Fields with Fractal and Hurst effects

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Abstract

In this document, additional results are provided to supplement *Lamb's Problem on Random Mass Density Fields with Fractal and Hurst effects*[1]

1 Introduction

This document is to serve as a supplement to *Lamb's Problem on Random Mass Density Fields with Fractal and Hurst effects* to be published in the Proceedings of the Royal Society of London, Series A (PRSA). The notation and terminology used here are consistent with the published article. Henceforth, we will refer to the published article as PRSA article.

This article is organized as follows: Section 2 contains additional results for Cauchy random fields (RFs) and serves to supplement the results from Section 5b of the PRSA article; Section 3 contains additional results for Dagum RFs to supplement Section 5c of the PRSA article.

2 Cauchy RF Response

For Cauchy RFs, we use the same coefficient of variation for the RF, $\mathrm{CV}_{RF} = 0.124$, as used before. Figure 1 and Figure 2, plot the response for $\alpha = 0.2$ and $\alpha = 1.0$, respectively, while varying β. From these figures, along with Figure 7 of the PRSA article, we note that as β increases, so

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[1]Nishawala VV, Ostoja-Starzewski M, Leamy MJ, Porcu E. (2016): Lamb's Problem on Random Mass Density Fields with Fractal and Hurst effects, Proceedings of the Royal Society of London, Series A, in print (the article linked to this supplement).

does SNR_R. However, as α increases, the affect of changing β is not as significant.

Figure 3, Figure 4 and Figure 5, plot the response for $\beta = 0.2$, $\beta = 1.0$ and $\beta = 1.8$, respectively, while varying α. From these figures we see that small values of α show the largest changes in response. The only exception to this trend is for small β's.

Figure 6a plots the mean and SD of the response of 128 realizations for $\alpha = 1.8$ and $\beta = 0.2$, for several values of CV_{RF}. Figures 6b and 6c plot the CV_R and SNR_R respectively. From the figures, we see that as CV_{RF} increases, CV_R increases, as expected. We also note that if $|\text{CV}_R| < \text{CV}_{RF}$ is satisfied for one CV_{RF}, then it is also satisfied for other CV_{RF}.

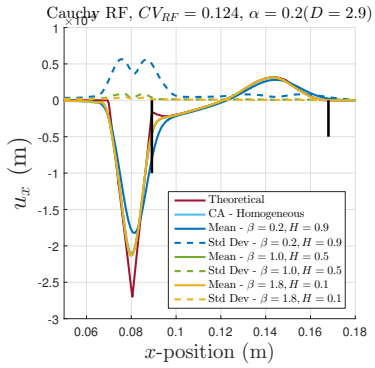
Figure 7 plots the results for over one-hundred values of α and β expanding the α, β space of Figure 8 of the PRSA article. Figure 7a plots the result for the pressure wave. A blue 'o' signals, for that point, the maximum SNR_R exceeded SNR_{RF} and a red 'x' denotes that the maximum SNR_R did not exceed SNR_{RF}. A dotted line is provided to show the approximate boarder between the two results. The same notation is used for the Rayleigh wave figure, Figure 7b.

3 Dagum RF Response

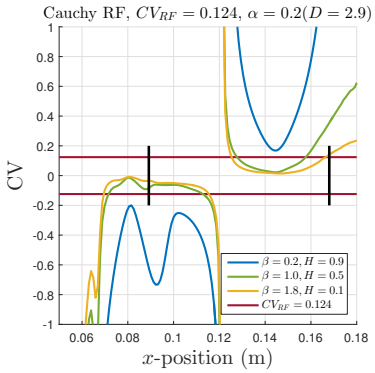
For Dagum RFs we use the same mean and CV_{RF} as in the Cauchy case. Figure 8 shows the nine different realizations of Dagum RF, changing α and β, that are among those used in our analysis. In Figure 8, the left column is $\beta = 0.2$ ($H = 0.9$), center column $\beta = 0.5$ ($H = 0.75$), and right column $\beta = 0.8$ ($H = 0.6$). The top row has $\alpha = 0.8$ ($D = 2.6$), the middle row $\alpha = 0.5$ ($D = 2.75$) and bottom row $\alpha = 0.2$ ($D = 2.9$). From the figures, similar to Cauchy RFs, we can clearly see the effect of changing fractal dimension and Hurst parameter. The figures toward the top-left have clear regions of high and low values, while the figures toward the bottom-right have smaller regions of high and low and small values, but it is not as apparent as Cauchy RFs.

Figures 9a, 10a and 11a plots the the response of 128 realizations for $\alpha = 0.2, 0.5$ and 0.8 respectively, while changing β. For each figure, the top most figure plots the mean and SD, the middle figure plots the CV_R and the bottom figure plots SNR_R. From the figures, we see that as β increases, CV_R decreases, similar to Cauchy RFs above. However, as α increases, the affect of changing β is not as pronounced. This is due to the changes in the RF, similar to what as we see with Cauchy RFs. For smaller values of β, like in Figure 8a, we see RFs with distinct areas of high and low values, similar to coarse WN RFs. For larger values of β, like in Figure 8c, the high and low values are better distributed, similar to fine WN RFs.

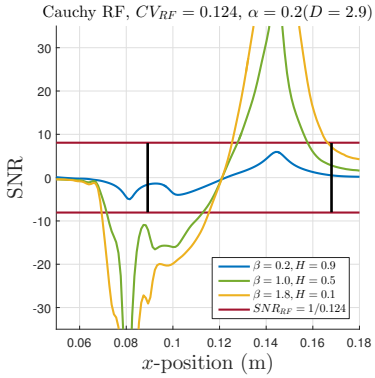
Figures 9, 10 and 11 plots the response of 128 realizations for $\alpha = 0.2, 0.5$ and 0.8 respectively, while changing β. For each figure, the top most figure plots the mean and SD, the middle figure plots the CV_R and the bottom figure plots SNR_R. From the figures, we see that as β increases, CV_R decreases, similar to Cauchy RFs above. However, as α



(a) Mean and SD

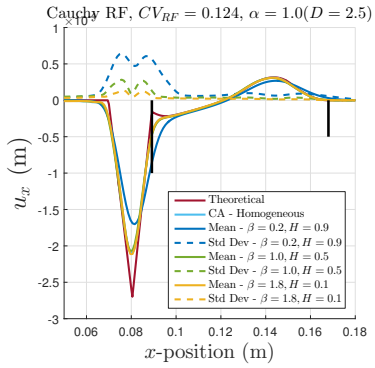


(b) CV

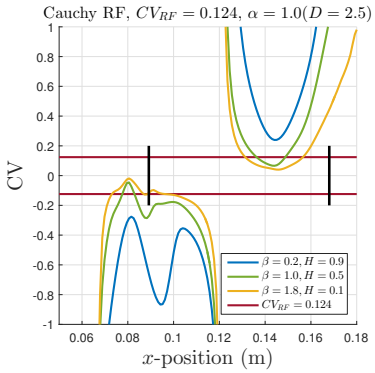


(c) SNR

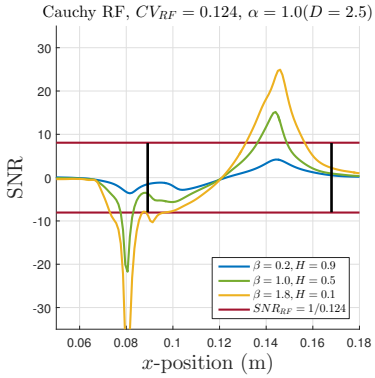
Figure 1: Cauchy RF with $\alpha = 0.2 (D = 2.9)$, varying β - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\beta = 0.2, H = 0.9$, Green line - $\beta = 1.0, H = 0.5$, Yellow line - $\beta = 1.8, H = 0.1$.



(a) Mean and SD

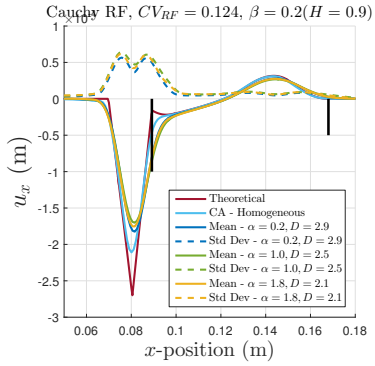


(b) CV

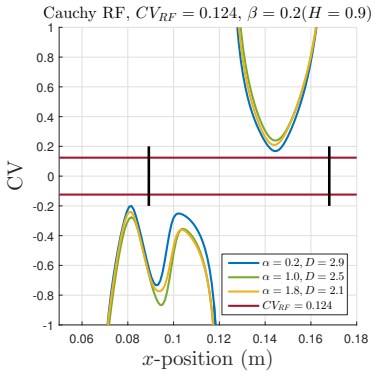


(c) SNR

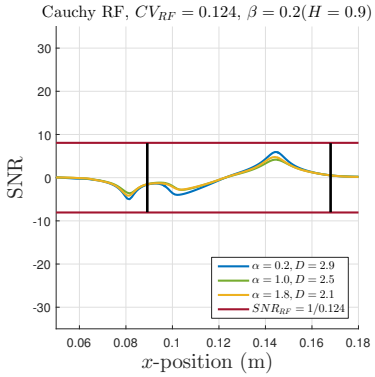
Figure 2: Cauchy RF with $\alpha = 1.0 (D = 2.5)$, varying β - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\beta = 0.2, H = 0.9$, Green line - $\beta = 1.0, H = 0.5$, Yellow line - $\beta = 1.8, H = 0.1$.



(a) Mean and SD

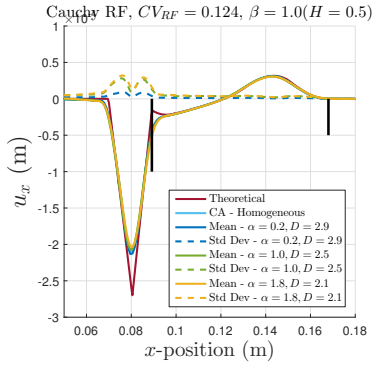


(b) CV

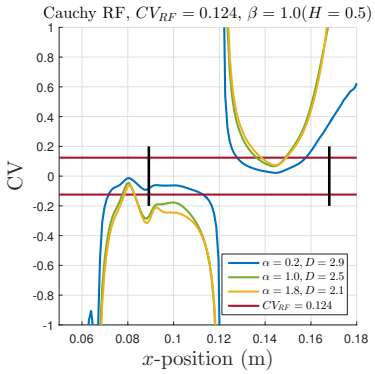


(c) SNR

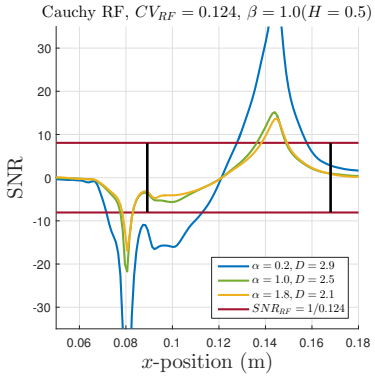
Figure 3: Cauchy RF with $\beta = 0.2(H = 0.9)$, varying α - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\alpha = 0.2, D = 2.9$, Green line - $\alpha = 1.0, D = 2.5$, Yellow line - $\alpha = 1.8, D = 2.1$.



(a) Mean and SD

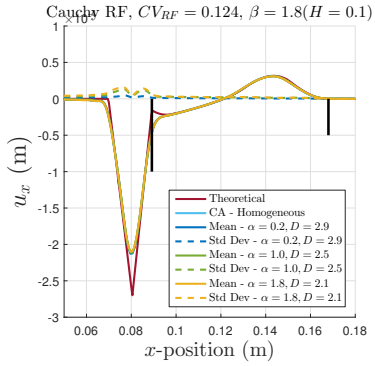


(b) CV

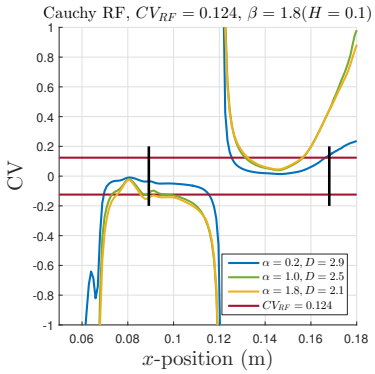


(c) SNR

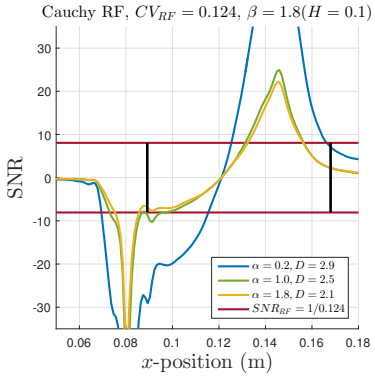
Figure 4: Cauchy RF with $\beta = 1.0(H = 0.5)$, varying α - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\alpha = 0.2, D = 2.9$, Green line - $\alpha = 1.0, D = 2.5$, Yellow line - $\alpha = 1.8, D = 2.1$.



(a) Mean and SD

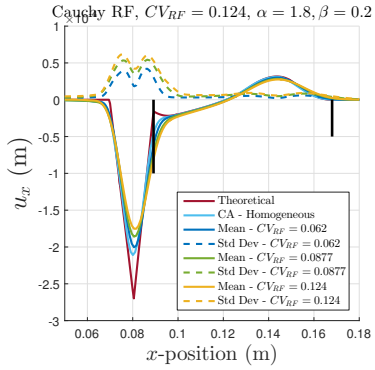


(b) CV

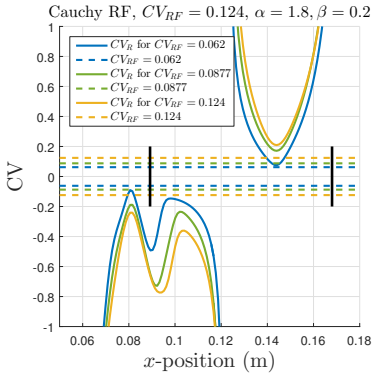


(c) SNR

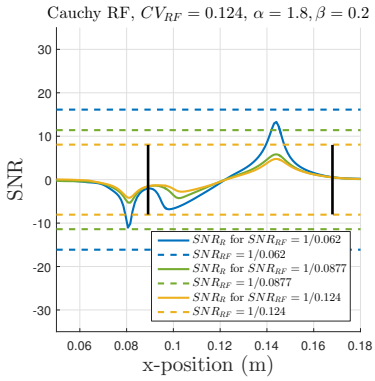
Figure 5: Cauchy RF with $\beta = 1.8(H = 0.1)$, varying α - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\alpha = 0.2, D = 2.9$, Green line - $\alpha = 1.0, D = 2.5$, Yellow line - $\alpha = 1.8, D = 2.1$.



(a) Mean and SD

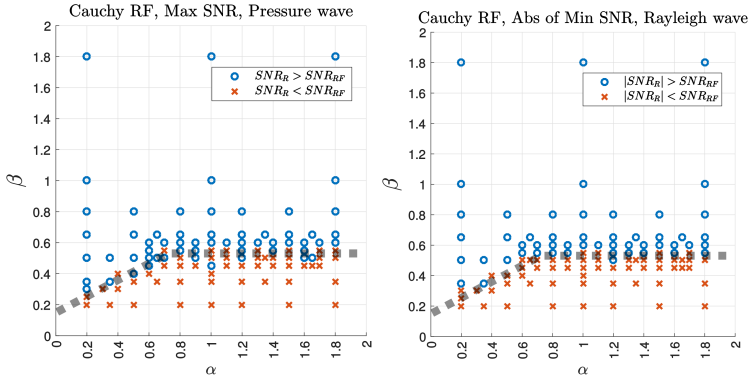


(b) CV



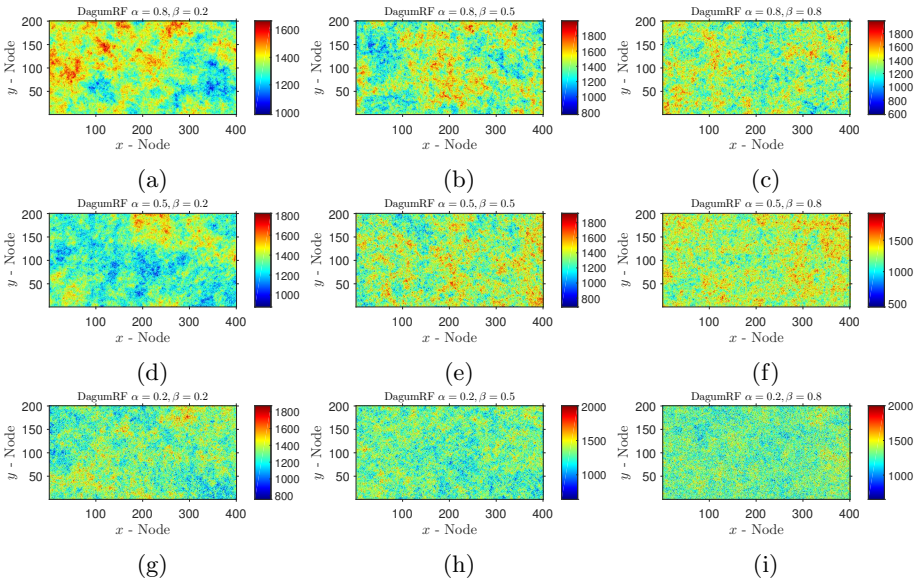
(c) SNR

Figure 6: Cauchy RF with $\alpha = 1.8, \beta = 0.2$, varying CV_{RF} - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\mathrm{CV}_{RF} = 0.062$, Green line - $\mathrm{CV}_{RF} = 0.0877$, Yellow line - $\mathrm{CV}_{RF} = 0.124$.



(a) (b)

Figure 7: Cauchy RF with $\text{CV}_{RF} = 0.124$, varying α and β. Dotted line denotes approximate boundary between 'greater than' and 'less than' results. (a) Maximum SNR_R compared to SNR_{RF} (b) Absolute value of minimum SNR_R compared to SNR_{RF}.

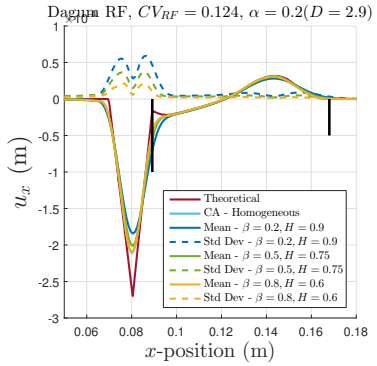


(a) (b) (c)

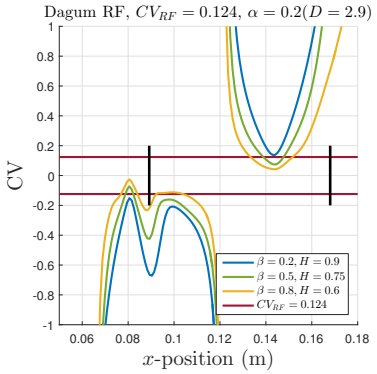
(d) (e) (f)

(g) (h) (i)

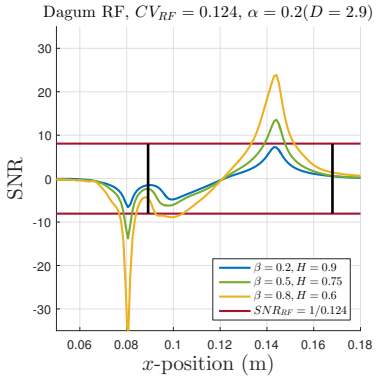
Figure 8: Dagum RF, $\text{CV}_{RF} = 0.124$, Left Column: $\beta = 0.2(H = 0.9)$, Center Column: $\beta = 0.5(H = 0.75)$, Right Column: $\beta = 0.8(H = 0.6)$. Top Row: $\alpha = 0.8(D = 2.6)$, Middle Row: $\alpha = 0.5(D = 2.75)$, Bottom Row: $\alpha = 0.2(D = 2.9)$. kg/m^3.



(a) Mean and SD

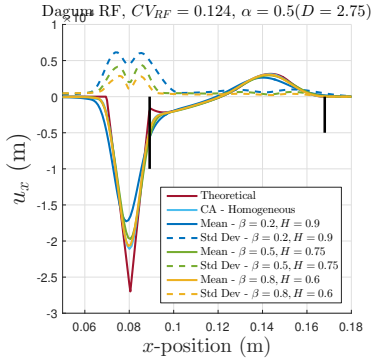


(b) CV

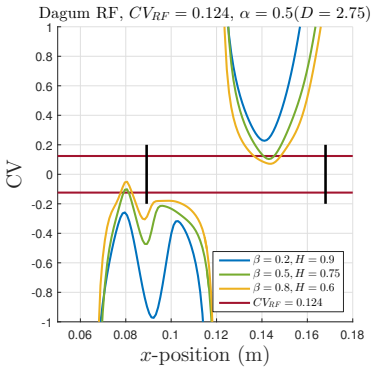


(c) SNR

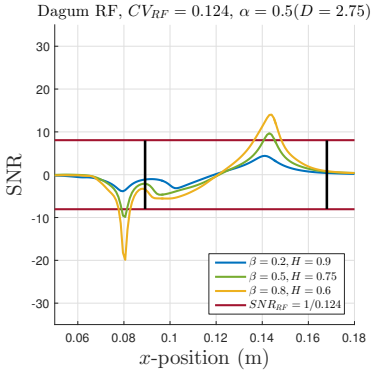
Figure 9: Dagum RF with $\alpha = 0.2 (D = 2.9)$, varying β - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\beta = 0.2, H = 0.9$, Green line - $\beta = 0.5, H = 0.75$, Yellow line - $\beta = 0.8, H = 0.6$.



(a) Mean and SD

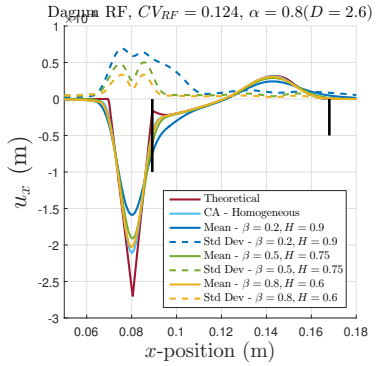


(b) CV

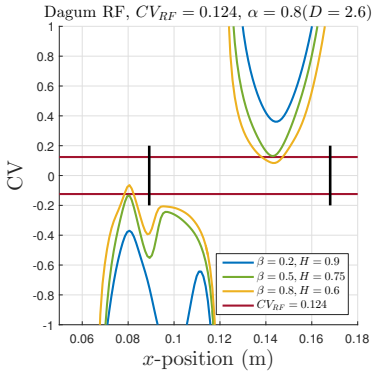


(c) SNR

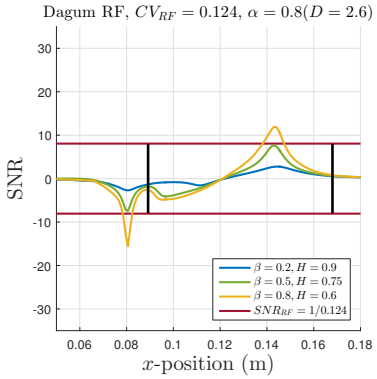
Figure 10: Dagum RF with $\alpha = 0.5(D = 2.75)$, varying β - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\beta = 0.2, H = 0.9$, Green line - $\beta = 0.5, H = 0.75$, Yellow line - $\beta = 0.8, H = 0.6$.



(a) Mean and SD



(b) CV



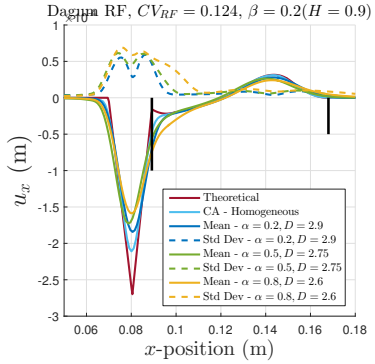
(c) SNR

Figure 11: Dagum RF with $\alpha = 0.8 (D = 2.6)$, varying β - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\beta = 0.2, H = 0.9$, Green line - $\beta = 0.5, H = 0.75$, Yellow line - $\beta = 0.8, H = 0.6$.

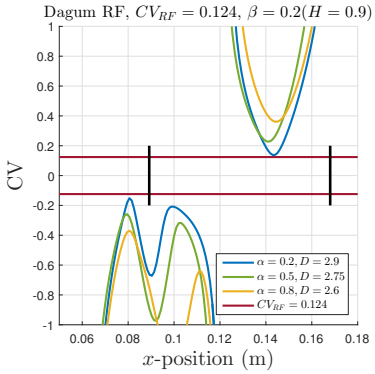
increases, the affect of changing β is not as pronounced. This is due to the changes in the RF, similar to what as we see with Cauchy RFs. For smaller values of β, like in Figure 8a, we see RFs with distinct areas of high and low values, similar to coarse WN RFs. For larger values of β, like in Figure 8c, the high and low values are better distributed, similar to fine WN RFs.

Figures 12, 13 and 14 plots the response of 128 realizations for $\beta = 0.2, 0.5$ and 0.8 respectively, while changing α. For each figure, plots of the mean, SD, CV_R and SNR_R are provided. Unfortunately, unlike above, we do not see a significant change in the results as we change α, or D. One interesting note, in Figure 12a there is a significant bump just in front of the Rayleigh wave at the 0.1 m mark. This bump is due to the Rayleigh wave being disrupted due to the randomness introduced. It is also unique in that we only see this bump for this α, β pair.

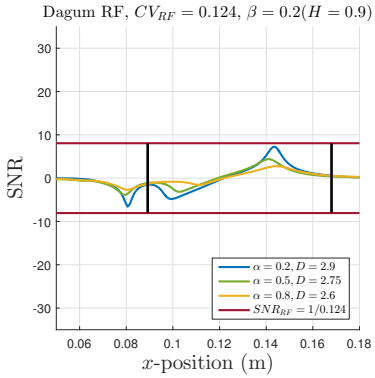
As stated in the PRSA manuscript, we tend to see more 'x's in Figures 10a and 12a than than in Figures 10b and 12b of the PRSA article. This could be due to P wave reflections and/or the elliptic motion of the R wave. An example of this unique R wave behavior can be seen in Figure 12a where there is a significant increase in the SD at the 0.1 m mark for $\alpha = 0.8, \beta = 0.2$. More analysis is needed to determine exactly why there is such a significant bump in the data.



(a) Mean and SD

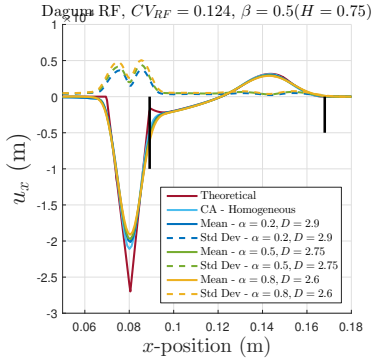


(b) CV

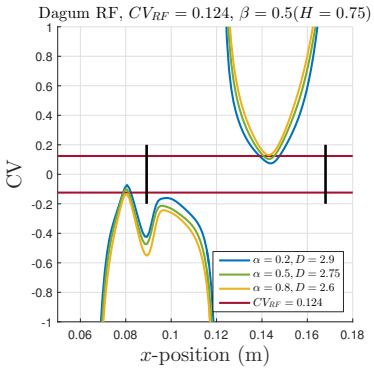


(c) SNR

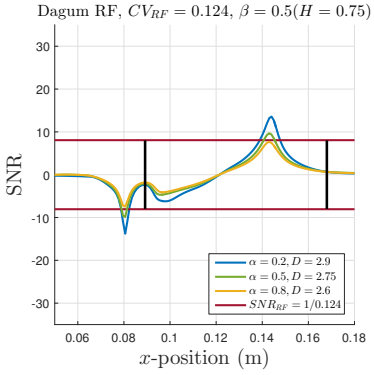
Figure 12: Dagum RF with $\beta = 0.2 (H = 0.9)$, varying α - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\alpha = 0.2, D = 2.9$, Green line - $\alpha = 0.5, D = 2.75$, Yellow line - $\alpha = 0.8, D = 2.6$.



(a) Mean and SD

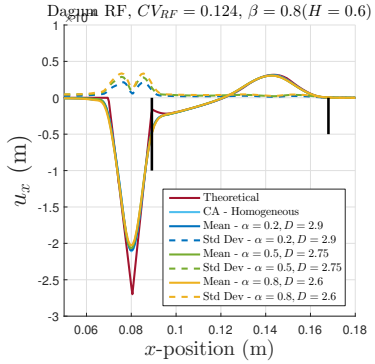


(b) CV

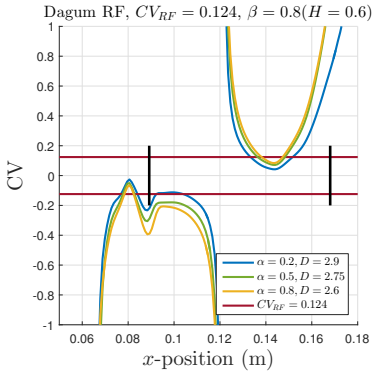


(c) SNR

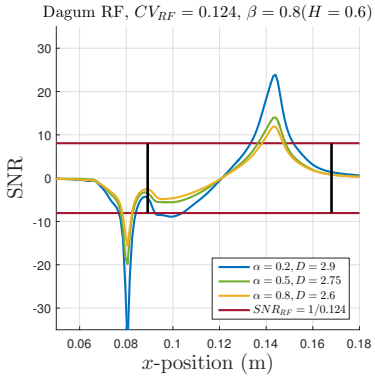
Figure 13: Dagum RF with $\beta = 0.5(H = 0.75)$, varying α - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\alpha = 0.2, D = 2.9$, Green line - $\alpha = 0.5, D = 2.75$, Yellow line - $\alpha = 0.8, D = 2.6$.



(a) Mean and SD



(b) CV



(c) SNR

Figure 14: Dagum RF with $\beta = 0.8(H = 0.6)$, varying α - (a) Theoretical solution and homogeneous results plotted with mean and standard deviations (b) CV of response (c) SNR of response, Blue line - $\alpha = 0.2, D = 2.9$, Green line - $\alpha = 0.5, D = 2.75$, Yellow line - $\alpha = 0.8, D = 2.6$.