Supplementary Information to Stability and Post-Bifurcation of Film-Substrate Systems


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Here, we provide the full details for the principle solution, first bifurcation, and the post-bifurcated solution derivation for the film-substrate system. We also include the results for a exponentially graded material.

# 1 Film-Substrate System

## 1.1 Problem Description

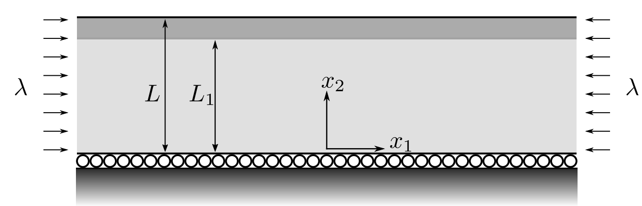


Figure 1: The geometry considered for the film-substrate problem, with infinite length in the $x_1$ direction and a finite thickness $L$ in the $x_2$ direction. We consider a piece-wise constant shear modulus with value $\mu_f$ in the top (film) layer, and $\mu_s$ in the bottom (substrate) layer. $\lambda$ is the prescribed average compressive strain.

Consider a two-dimensional (2D) strip of layered material with infinite length and finite thickness, occupying region $\Omega = (-\infty, \infty) \times (0, L)$, as shown in Figure 1. The film thickness is defined as $t = L - L_1$. Here, it is assumed the film is perfectly bonded to the substrate. We consider the case of displacement control, where the loading parameter $\lambda$ prescribes the average stretch in the $x_1$ direction, $\lambda_1 = 1 - \lambda$. We will adopt a 2D plane strain, compressible Neo-Hookean constitutive law with internal energy density function,

$$W(F; x_2) = \mu(x_2) \left[ \frac{1}{2} (I_1 - 2 - \log I_2) + \frac{\nu}{1-\nu} \left( \sqrt{I_2} - 1 \right)^2 \right] = \mu(x_2) \, \tilde{W}(F), \tag{1.1.1}$$

where $F$ is the deformation gradient, and $I_1$ and $I_2$ are the 2D invariants of the Cauchy–Green tensor $C = F^T F$. These are given by

$$I_1 = \operatorname{tr} C, \quad I_2 = \det C. \tag{1.1.2}$$

The shear modulus $\mu(x_2)$ is taken as a piece-wise function, having constant values in the film and substrate,

$$\mu(x_2) = \begin{cases} \mu_s, & x_2 \in [0, L_1) \\ \mu_f, & x_2 \in [L_1, L] \end{cases}. \tag{1.1.3}$$

For simplicity, we assume that both the film and substrate share the same Poisson ratio $\nu$.

The total strain energy for a given displacement field $u(x) : \Omega \to \mathbb{R}^2$, can be expressed as[1],

$$\mathcal{E}(u; \lambda) = \int_\Omega W(\nabla u; x_2)\, d\Omega. \tag{1.1.4}$$

The first variation gives the equilibrium condition for a displacement field,

$$\mathcal{E}_{,u}(u; \lambda)\, \delta u = \int_\Omega \frac{\partial W}{\partial F} \nabla \delta u\, d\Omega = 0 \qquad \forall \delta u \in KA, \tag{1.1.5}$$

where $KA$ is the space of kinematically admissible displacement variations, having average strain $\lambda$ in the $x_1$ direction, zero vertical displacement at $x_2 = 0$, and zero average horizontal displacement at $x_2 = 0$.

## 1.2 Principal Solution

Here, we solve for the principal solution, $\overset{0}{u}$, which satisfies equilibrium, passes though the undeformed configuration, and remains stable for small $\lambda$. The first Piola-Kirchoff stress can be expressed as,

$$\Pi = \frac{\partial W}{\partial F} = \mu \left(F - F^{-T}\right) + \frac{2\nu\mu}{1-\nu} \left(I^2 - \sqrt{I_2}\right) F^{-T}. \tag{1.2.1}$$

We assume a deformation with constant deformation gradient $\overset{0}{F} = \mathrm{diag}(\lambda_1, \lambda_2)$, expressed in the Cartesian basis aligned with the $x_1$ and $x_2$ axis. Here $\lambda_1$ is assumed known in terms of the loading $\lambda$, i.e., $\lambda_1 := 1 - \lambda$. Using this, we have,

$$\Pi|_{\overset{0}{F}} = \mathrm{diag}(\Pi_{11}, \Pi_{22}), \tag{1.2.2}$$

where,

$$\begin{aligned} \Pi_{11} &= \mu \left[\lambda_1 - \lambda_1^{-1} + \frac{2\nu}{1-\nu}(\lambda_1\lambda_2^2 - \lambda_2)\right], \\ \Pi_{22} &= \mu \left[\lambda_2 - \lambda_2^{-1} + \frac{2\nu}{1-\nu}(\lambda_1^2\lambda_2 - \lambda_1)\right]. \end{aligned} \tag{1.2.3}$$

Enforcing zero normal traction on the free surface gives,

$$\Pi_{22}|_{x_2=L} = 0 \qquad \Longrightarrow \qquad \lambda_2 - \lambda_2^{-1} + \frac{2\nu}{1-\nu}(\lambda_1^2\lambda_2 - \lambda_1) = 0. \tag{1.2.4}$$

Thus, the unknown vertical stretch $\lambda_2$ is the solution to the quadratic,

$$\lambda_2^2 - 1 + \frac{2\nu}{1-\nu}(\lambda_1^2\lambda_2^2 - \lambda_1\lambda_2) = 0. \tag{1.2.5}$$

We consider only the non-negative root of this quadratic. For $0 < \nu < 1$ this is,

$$\lambda_2 = \frac{1}{2\left(\frac{2\nu}{1-\nu}\lambda_1^2 + 1\right)} \left[\frac{2\nu}{1-\nu}\lambda_1 + \sqrt{\left(\frac{2\nu}{1-\nu}\right)^2 \lambda_1^2 + 4\left(\frac{2\nu}{1-\nu}\lambda_1^2 + 1\right)}\right]. \tag{1.2.6}$$

This homogeneous solution satisfies the equilibrium equation in (1.1.5). It also passes through the undeformed configuration at $\lambda = 0$, as then $\lambda_1 = \lambda_2 = 1$. Stability along this principal solution can be analyzed with the help of the incremental moduli,

$$L_{ijkl} := \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} = \mu \delta_{ik}\delta_{jl} + \left[\mu - \frac{2\nu\mu}{1-\nu}\left(I_2 - \sqrt{I_2}\right)\right] F_{jk}^{-1} F_{li}^{-1} + \frac{2\mu\nu}{1-\nu}\left(2I_2 - \sqrt{I_2}\right) F_{lk}^{-1} F_{ji}^{-1}. \tag{1.2.7}$$

[1]For practical purposes, we implicitly replace the infinite domain with a periodic domain of periodic length as large as desired.

Then, along the principal solution, the non-zero components of the incremental moduli may be expressed as,

$$\begin{aligned}
L^0_{1111} &= \frac{\mu}{\lambda_1^2(1-\nu)}\left[\lambda_1^2\left(2\nu\lambda_2^2+1-\nu\right)+1-\nu\right],\\
L^0_{2222} &= \frac{\mu}{\lambda_2^2(1-\nu)}\left[\lambda_2^2\left(2\nu\lambda_1^2+1-\nu\right)+1-\nu\right],\\
L^0_{1122} &= L^0_{2211} = \frac{2\mu\nu}{1-\nu}\left(2\lambda_1\lambda_2-1\right),\\
L^0_{1212} &= L^0_{2121} = \mu,\\
L^0_{1221} &= L^0_{2112} = \frac{\mu}{\lambda_1\lambda_2(1-\nu)}\left[2\nu\left(\lambda_1\lambda_2-\lambda_1^2\lambda_2^2\right)+1-\nu\right] = \mu\frac{\lambda_2}{\lambda_1},
\end{aligned} \tag{1.2.8}$$

where $L^0_{ijkl} = \left.\frac{\partial^2 W}{\partial F_{ij} F_{kl}}\right|_{\overset{0}{F}(\lambda)}$.

## 1.3 First Bifurcated Solution

As the solid continues to be compressed, it will reach a point when the principle solution loses stability. This transition occurs when the second variation of the energy becomes singular,

$$\left(\mathcal{E}_{,uu}(\overset{0}{u}(\lambda);\lambda)\,\overset{1}{u}\right)\delta u = \int_\Omega \left(\nabla\delta u \left.\frac{\partial^2 W}{\partial F\partial F}\right|_{\overset{0}{F}(\lambda)} \nabla\overset{1}{u}\right) d\Omega = 0, \qquad \forall \delta u \in KA, \tag{1.3.1}$$

for some bifurcated mode, $\overset{1}{u} \in KA$. Considering a Cartesian basis aligned with the $x_1$ and $x_2$ axis, this can be written as,

$$\int_\Omega \mu(x_2)\tilde{L}^0_{ijkl}\overset{1}{u}_{k,l}\delta u_{i,j}\, d\Omega = 0, \qquad \forall\delta u \in KA, \tag{1.3.2}$$

where $\tilde{L}^0_{ijkl} = \left.\frac{\partial^2 \tilde{W}}{\partial F_{ij} F_{kl}}\right|_{\overset{0}{F}(\lambda)}$, and we use the notation $\delta u_{i,j} := \frac{\partial \delta u_i}{\partial x_j}$. We consider $\overset{1}{u}$ to be a piece-wise function of the film and substrate as,

$$\overset{1}{u}(x_1,x_2) = \begin{cases} \overset{1_s}{u}, & x_2 \in [0, L_1) \\ \overset{1_f}{u}, & x_2 \in [L_1, L] \end{cases}. \tag{1.3.3}$$

Then, splitting the integrals in (1.3.2) into domains for the film and substrate, and applying the Gauss divergence theorem gives,

$$\begin{aligned}
0 = &\mu_s \int_{\partial\Omega_s} \tilde{L}^0_{ijkl}\overset{1_s}{u}_{k,l} N_j \delta u_i\, ds - \mu_s \int_{\Omega_s} \tilde{L}^0_{ijkl}\overset{1_s}{u}_{k,lj}\delta u_i\, d\Omega \\
&+ \mu_f \int_{\partial\Omega_f} \tilde{L}^0_{ijkl}\overset{1_f}{u}_{k,l} N_j \delta u_i\, ds - \mu_f \int_{\Omega_f} \tilde{L}^0_{ijkl}\overset{1_f}{u}_{k,lj}\delta u_i\, d\Omega,
\end{aligned} \tag{1.3.4}$$

where $N$ is the reference outward unit normal vector. Localizing gives the linear second-order system of PDE's that holds for both the film and substrate domains,

$$\tilde{L}^0_{ijkl}\,\overset{1}{u}_{k,lj} = 0, \qquad i = 1, 2. \tag{1.3.5}$$

Using the major symmetries and known zero components of the incremental moduli, this can be expanded to,

$$\begin{aligned}
&\tilde{L}^0_{1111}\overset{1}{u}_{1,11} + \tilde{L}^0_{1212}\overset{1}{u}_{1,22} + \tilde{L}^0_{1221}\overset{1}{u}_{2,12} + \tilde{L}^0_{1122}\overset{1}{u}_{2,21} = 0,\\
&\tilde{L}^0_{2222}\overset{1}{u}_{2,22} + \tilde{L}^0_{2121}\overset{1}{u}_{2,11} + \tilde{L}^0_{2112}\overset{1}{u}_{1,21} + \tilde{L}^0_{2211}\overset{1}{u}_{1,12} = 0.
\end{aligned} \tag{1.3.6}$$

The integrals over the external boundaries and the film-substrate interface give the natural boundary conditions. The shear-free and vertical displacement conditions on the bottom edge are,

$$\tilde{L}^0_{12kl} \overset{1_s}{u}_{k,l}(x_1, 0) = 0, \qquad \overset{1_s}{u}_2(x_1, 0) = 0. \tag{1.3.7}$$

The traction-free conditions on the free surface result in,

$$\tilde{L}^0_{i2kl} \overset{1_f}{u}_{k,l}(x_1, L) = 0, \qquad i = 1, 2. \tag{1.3.8}$$

The film-substrate interface conditions for traction and displacement continuity gives,

$$\tilde{L}^0_{i2kl} \overset{1_s}{u}_{k,l}(x_1, L_1) = \frac{\mu_f}{\mu_s} \tilde{L}^0_{i2kl} \overset{1_f}{u}_{k,l}(x_1, L_1), \quad i = 1, 2, \qquad \overset{1_s}{u}(x_1, L_1) = \overset{1_f}{u}(x_1, L_1). \tag{1.3.9}$$

Finally, the zero average horizontal displacement condition at $x_2 = 0$ gives

$$\int \overset{1_s}{u}_1(x_1, 0) dx_1 = 0. \tag{1.3.10}$$

We now look to separate the PDE's in (1.3.5). We assume each of the vector components in $\overset{1}{u}$ may be separated as,

$$\overset{1}{u}_1 = X_1(x_1) Y_1(x_2), \qquad \overset{1}{u}_2 = X_2(x_1) Y_2(x_2). \tag{1.3.11}$$

Substituting this into each of the PDE's in (1.3.6) gives,

$$\begin{aligned} &\tilde{L}^0_{1111} X_1'' Y_1 + \tilde{L}^0_{1212} X_1 Y_1'' + \left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right) X_2' Y_2' = 0, \\ &\tilde{L}^0_{2222} X_2 Y_2'' + \tilde{L}^0_{2121} X_2'' Y_2 + \left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) X_1' Y_1' = 0, \end{aligned} \tag{1.3.12}$$

where the prime denotes a derivative with the function's dependent variable. For non-zero $X_1$, $X_2$, $Y_1$, $Y_2$,

$$\begin{aligned} &\tilde{L}^0_{1111} \frac{X_1''}{X_1} + \tilde{L}^0_{1212} \frac{Y_1''}{Y_1} + \left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right) \frac{X_2'}{X_1} \frac{Y_2'}{Y_1} = 0, \\ &\tilde{L}^0_{2222} \frac{Y_2''}{Y_2} + \tilde{L}^0_{2121} \frac{X_2''}{X_2} + \left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) \frac{X_1'}{X_2} \frac{Y_1'}{Y_2} = 0. \end{aligned} \tag{1.3.13}$$

Taking derivatives with $x_1$ and $x_2$ of both equations yields,

$$\begin{aligned} &\left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right) \left(\frac{X_2'}{X_1}\right)' \left(\frac{Y_2'}{Y_1}\right)' = 0, \\ &\left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) \left(\frac{X_1'}{X_2}\right)' \left(\frac{Y_1'}{Y_2}\right)' = 0, \end{aligned} \tag{1.3.14}$$

requiring

$$\begin{aligned} &\frac{X_2'}{X_1} = \text{const} \quad \text{or} \quad \frac{Y_2'}{Y_1} = \text{const}, \\ &\text{and} \\ &\frac{X_1'}{X_2} = \text{const} \quad \text{or} \quad \frac{Y_1'}{Y_2} = \text{const}. \end{aligned} \tag{1.3.15}$$

We find that only one of these four cases results in bounded, non-trivial solutions that satisfy the boundary and zero average strain conditions. For scalar constants $c_1$ and $c_2$ this is,

$$\frac{X_2'}{X_1} = c_1, \quad \frac{X_1'}{X_2} = c_2. \tag{1.3.16}$$

These can be used to separate (1.3.13), resulting in the following set of ODE's,

$$\begin{aligned} X_2' &= c_1 X_1, & X_1'' &= \frac{c_3}{\tilde{L}^0_{1111}} X_1, & \tilde{L}^0_{1212} Y_1'' + \left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right) c_1 Y_2' + c_3 Y_1 = 0, \\ X_1' &= c_2 X_2, & X_2'' &= \frac{c_4}{\tilde{L}^0_{2121}} X_2, & \tilde{L}^0_{2222} Y_2'' + \left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) c_2 Y_1' + c_4 Y_2 = 0, \end{aligned} \tag{1.3.17}$$

where $c_3$ and $c_4$ are additional constants that arise from the separation. This system can be un-coupled, resulting in the following set of ODEs,

$$\begin{aligned} X_1'' + \omega^2 X_1 &= 0, \\ X_2'' + \omega^2 X_2 &= 0, \\ \tilde{L}^0_{2222}\tilde{L}^0_{1212} Y_1'''' + \omega^2 \left[\left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right)\left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) - \tilde{L}^0_{2112}\tilde{L}^0_{1212} - \tilde{L}^0_{1111}\tilde{L}^0_{2222}\right] Y_1'' + \omega^4 \tilde{L}^0_{1111}\tilde{L}^0_{2112} Y_1 &= 0, \\ \tilde{L}^0_{2222}\tilde{L}^0_{1212} Y_2'''' + \omega^2 \left[\left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right)\left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) - \tilde{L}^0_{2112}\tilde{L}^0_{1212} - \tilde{L}^0_{1111}\tilde{L}^0_{2222}\right] Y_2'' + \omega^4 \tilde{L}^0_{1111}\tilde{L}^0_{2112} Y_2 &= 0, \end{aligned} \tag{1.3.18}$$

where $\omega$ is the only independent separation constant (in particular, it is found that $-\omega^2 := c_1 c_2 = c_3/\tilde{L}^0_{1111} = c_4/\tilde{L}^0_{2112}$).

Solving the first two equations of (1.3.18), and using the additional conditions in the first column of (1.3.17), we find two linearly independent solutions for the $x_1$ part of the solution. These may be written as anti-symmetric (corresponding to $c_1 = -\omega$ and $c_2 = \omega$) and symmetric (corresponding to $c_1 = \omega$ and $c_2 = -\omega$) about $x_1 = 0$ solutions

$$\begin{aligned} &\mathcal{A} : X_1(x_1) = \sin(\omega x_1), \quad X_2(x_1) = \cos(\omega x_1), \\ &\text{and} \\ &\mathcal{S} : X_1(x_1) = \cos(\omega x_1), \quad X_2(x_1) = \sin(\omega x_1). \end{aligned} \tag{1.3.19}$$

Next, solving the last two equations of (1.3.18), using the additional conditions in the last column of (1.3.17), we find four linearly independent exponential solutions for the $x_2$ part of the solution. These may be written as

$$\begin{aligned} &\mathcal{A} : \begin{cases} Y_1(x_2) = e^{\alpha_1 x_2}, \quad Y_2(x_2) = -B_1 e^{\alpha_1 x_2}, \\ Y_1(x_2) = e^{\alpha_2 x_2}, \quad Y_2(x_2) = -B_2 e^{\alpha_2 x_2}, \\ Y_1(x_2) = e^{\alpha_3 x_2}, \quad Y_2(x_2) = -B_3 e^{\alpha_3 x_2}, \\ Y_1(x_2) = e^{\alpha_4 x_2}, \quad Y_2(x_2) = -B_4 e^{\alpha_4 x_2}; \end{cases} \\ &\text{and} \\ &\mathcal{S} : \begin{cases} Y_1(x_2) = e^{\alpha_1 x_2}, \quad Y_2(x_2) = B_1 e^{\alpha_1 x_2}, \\ Y_1(x_2) = e^{\alpha_2 x_2}, \quad Y_2(x_2) = B_2 e^{\alpha_2 x_2}, \\ Y_1(x_2) = e^{\alpha_3 x_2}, \quad Y_2(x_2) = B_3 e^{\alpha_3 x_2}, \\ Y_1(x_2) = e^{\alpha_4 x_2}, \quad Y_2(x_2) = B_4 e^{\alpha_4 x_2}. \end{cases} \end{aligned} \tag{1.3.20}$$

In the above, the $\alpha$'s are roots of the characteristic equation for the last two ODEs in (1.3.18)

$$a\alpha^4 + b\alpha^2 + c = 0,$$

$$\begin{aligned} a &= \tilde{L}^0_{2222}\tilde{L}^0_{1212}, \\ b &= \omega^2 \left[\left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right)\left(\tilde{L}^0_{2112} + \tilde{L}^0_{2211}\right) - \tilde{L}^0_{2112}\tilde{L}^0_{1212} - \tilde{L}^0_{1111}\tilde{L}^0_{2222}\right], \\ c &= \omega^4 \tilde{L}^0_{1111}\tilde{L}^0_{2112}, \end{aligned}$$

and the $B$'s are found (from the last column of (1.3.17)) to be given by

$$B_i = \frac{\omega^2 \tilde{L}^0_{1111} - \tilde{L}^0_{1212}\alpha_i^2}{\omega\left(\tilde{L}^0_{1221} + \tilde{L}^0_{1122}\right)\alpha_i}. \tag{1.3.21}$$

Taking arbitrary linear combinations of the $Y$'s (by introducing the undetermined constants $A_j$ and defining $v_1(x_2) := \sum_{i=1}^{4} A_i e^{\alpha_i x_2}$ and $v_2(x_2) := \sum_{i=1}^{4} A_i B_i e^{\alpha_i x_2}$, we obtain the following anti-symmetric and symmetric solutions to (1.3.8)

$$
\begin{aligned}
&\mathcal{A} : \begin{cases} \overset{1}{u}_1(x_1, x_2) = \sin(\omega x_1) v_1(x_2), \\ \overset{1}{u}_2(x_1, x_2) = \cos(\omega x_1)[-v_2(x_2)]; \end{cases} \\
&\mathcal{S} : \begin{cases} \overset{1}{u}_1(x_1, x_2) = \cos(\omega x_1) v_1(x_2), \\ \overset{1}{u}_2(x_1, x_2) = \sin(\omega x_1) v_2(x_2). \end{cases}
\end{aligned}
\tag{1.3.22}
$$

Finally, we recall that these solutions are valid in both the film and substrate domains. So, to express the complete piece-wise general form of the anti-symmetric and symmetric solutions we reintroduce the superscripts $s$ and $f$. Further, we multiply the entire asymmetric solution by minus one for convenience in the derivation and write

$$
\begin{aligned}
&\mathcal{A} : \begin{cases} \overset{1}{u}_1(x_1, x_2) = \begin{cases} \overset{1_s}{u}_1 = -\sin(\omega x_1) v_1^s(x_2), & x_2 \in [0, L_1), \\ \overset{1_f}{u}_1 = -\sin(\omega x_1) v_1^f(x_2), & x_2 \in [L_1, L], \end{cases} \\ \overset{1}{u}_2(x_1, x_2) = \begin{cases} \overset{1_s}{u}_2 = \cos(\omega x_1) v_2^s(x_2), & x_2 \in [0, L_1), \\ \overset{1_f}{u}_2 = \cos(\omega x_1) v_2^f(x_2), & x_2 \in [L_1, L], \end{cases} \end{cases} \\
&\mathcal{S} : \begin{cases} \overset{1}{u}_1(x_1, x_2) = \begin{cases} \overset{1_s}{u}_1 = \cos(\omega x_1) v_1^s(x_2), & x_2 \in [0, L_1), \\ \overset{1_f}{u}_1 = \cos(\omega x_1) v_1^f(x_2), & x_2 \in [L_1, L], \end{cases} \\ \overset{1}{u}_2(x_1, x_2) = \begin{cases} \overset{1_s}{u}_2 = \sin(\omega x_1) v_2^s(x_2), & x_2 \in [0, L_1), \\ \overset{1_f}{u}_2 = \sin(\omega x_1) v_2^f(x_2), & x_2 \in [L_1, L], \end{cases} \end{cases}
\end{aligned}
\tag{1.3.23}
$$

where $v_1^{s|f}(x_2) := \sum_{i=1}^{4} A_i^{s|f} e^{\alpha_i x_2}$ and $v_2^{s|f}(x_2) := \sum_{i=1}^{4} A_i^{s|f} B_i e^{\alpha_i x_2}$, giving a total of eight undertermined coefficients: $A_i^s$ and $A_i^f$ for $i = 1, \ldots, 4$.

Inserting either (anti-symmetric or symmetric) form of our solution into the eight boundary conditions

(1.3.7)–(1.3.9) gives a linear, homogeneous set of equations for the eight unknowns $A_i^s$ and $A_i^f$,

$$
\begin{aligned}
\sum_{i=1}^{4}\left[\tilde{L}^0_{1212}\alpha_i A_i^s + \omega \tilde{L}^0_{1221}A_i^s B_i\right] &= 0,\\
\sum_{i=1}^{4} A_i^s B_i &= 0,\\
\sum_{i=1}^{4}\left[\tilde{L}^0_{1212}\alpha_i A_i^f + \omega \tilde{L}^0_{1221}A_i^f B_i\right] e^{\alpha_i L} &= 0,\\
\sum_{i=1}^{4}\left[\tilde{L}^0_{2222}\alpha_i A_i^f B_i - \omega \tilde{L}^0_{2211}A_i^f\right] e^{\alpha_i L} &= 0,\\
\sum_{i=1}^{4}\left[\tilde{L}^0_{1212}\alpha_i A_i^s + \omega \tilde{L}^0_{1221}A_i^s B_i\right] e^{\alpha_i L_1} - \frac{\mu_f}{\mu_s}\sum_{i=1}^{4}\left[\tilde{L}^0_{1212}\alpha_i A_i^f + \omega \tilde{L}^0_{1221}A_i^f B_i\right] e^{\alpha_i L_1} &= 0,\\
\sum_{i=1}^{4}\left[\tilde{L}^0_{2222}\alpha_i A_i^s B_i - \omega \tilde{L}^0_{2211}A_i^s\right] e^{\alpha_i L_1} - \frac{\mu_f}{\mu_s}\sum_{i=1}^{4}\left[\tilde{L}^0_{2222}\alpha_i A_i^f B_i - \omega \tilde{L}^0_{2211}A_i^f\right] e^{\alpha_i L_1} &= 0,\\
\sum_{i=1}^{4} A_i^s e^{\alpha_i L_1} - \sum_{i=1}^{4} A_i^f e^{\alpha_i L_1} &= 0,\\
\sum_{i=1}^{4} A_i^s B_i e^{\alpha_i L_1} - \sum_{i=1}^{4} A_i^f B_i e^{\alpha_i L_1} &= 0.
\end{aligned}
\tag{1.3.24}
$$

The first two lines are the shear-free and vertical displacement conditions on the bottom edge from (1.3.7). The third and fourth line are the zero traction conditions at the free surface from (1.3.8). The fifth and sixth lines are the traction continuity conditions at the interface. Finally, the last two lines are the displacement continuity at the interface from (1.3.9). This system can be expressed in matrix form,

$$\mathbf{MA} = \mathbf{0}, \tag{1.3.25}$$

where $\mathbf{A} = [A_1^s, \dots, A_4^s, A_1^f, \dots, A_4^f]^T$ is the vector of amplitudes, and $\mathbf{M}$ is the assembled matrix of coefficients from the boundary conditions. Then our first bifurcated solution occurs at the smallest $\lambda$ that satisfies,

$$\det(\mathbf{M}) = 0. \tag{1.3.26}$$

Because the incremental moduli depend on $\lambda$, the matrix $\mathbf{M}$ depends on $\lambda$. Then, for a given $\omega$, we may solve for the smallest $\lambda$ that satisfies (1.3.26). This gives the load for modes of that frequency to occur, and we denote this as $\hat{\lambda}(\omega)$. The smallest of these over all wavelengths gives the critical load, $\lambda_c$, to trigger instability for the first mode,

$$\lambda_c = \min_{\omega}\ \hat{\lambda}(\omega), \qquad \omega_c = \arg\min_{\omega}\ \hat{\lambda}(\omega). \tag{1.3.27}$$

# 2 Asymptotic Mode

We look to solve for the unknown field $\overset{2}{u}$ at the critical load $\lambda_c$ which has critical frequency $\omega_c$. This can be written as (c.f., (2.3.10) in the main article),

$$\int_\Omega \left(\mu(x_2)\tilde{L}^c_{ijkl}\overset{2}{u}_{k,l} + \mu(x_2)\tilde{M}^c_{ijklmn}\overset{1}{u}_{k,l}\overset{1}{u}_{m,n}\right)\delta v_{i,j}\, d\Omega = 0, \qquad \forall \delta v \in \mathcal{N}^\perp, \tag{2.0.1}$$

where $\tilde{M}^c_{ijklmn} = \frac{\partial^3 \tilde{W}}{\partial F_{ij}\partial F_{kl}\partial F_{mn}}\Big|_{\overset{0}{F}(\lambda_c)}$ and $\tilde{L}^c_{ijkl} = \frac{\partial^2 \tilde{W}}{\partial F_{ij}\partial F_{kl}}\Big|_{\overset{0}{F}(\lambda_c)}$. We assume a piece-wise solution for $\overset{2}{u}$,

$$\overset{2}{u}(x_1, x_2) = \begin{cases} \overset{2_s}{u}, & x_2 \in [0, L_1) \\ \overset{2_f}{u}, & x_2 \in [L_1, L] \end{cases}. \tag{2.0.2}$$

Splitting the integral in (2.0.1) into the film and substrate domains and applying the Gauss divergence theorem gives,

$$\begin{aligned} 0 =& \mu_s \int_{\partial\Omega_s} \left( \tilde{L}^c_{ijkl} \overset{2_s}{u}_{k,l} + \tilde{M}^c_{ijklmn} \overset{1_s}{u}_{k,l} \overset{1_s}{u}_{m,n} \right) n_j \delta v_i ds \\ &- \mu_s \int_{\Omega_s} \left( \tilde{L}^c_{ijkl} \overset{2_s}{u}_{k,l} + \tilde{M}^c_{ijklmn} \overset{1_s}{u}_{k,l} \overset{1_s}{u}_{m,n} \right)_{,j} \delta v_i d\Omega \\ &+ \mu_f \int_{\partial\Omega_f} \left( \tilde{L}^c_{ijkl} \overset{2_f}{u}_{k,l} + \tilde{M}^c_{ijklmn} \overset{1_f}{u}_{k,l} \overset{1_f}{u}_{m,n} \right) n_j \delta v_i ds \\ &- \mu_2 \int_{\Omega_f} \left( \tilde{L}^c_{ijkl} \overset{2_f}{u}_{k,l} + \tilde{M}^c_{ijklmn} \overset{1_f}{u}_{k,l} \overset{1_f}{u}_{m,n} \right)_{,j} \delta v_i d\Omega. \end{aligned} \tag{2.0.3}$$

and localizing gives the governing system of PDE's,

$$\tilde{L}^c_{ijkl} \overset{2}{u}_{k,lj} = - \left( \tilde{M}^c_{ijklmn} \overset{1}{u}_{k,l} \overset{1}{u}_{m,n} \right)_{,j}, \qquad \text{for } i = 1, 2, \tag{2.0.4}$$

which holds in both the film and substrate domains. It should be noted that this is the same PDE that appears for the first bifurcation in (1.3.6), except now it is inhomogeneous, and we must restrict outselves to solutions in $N^{\perp}$. The boundary conditions on the bottom edge are,

$$\tilde{L}^c_{12kl} \overset{2_s}{u}_{k,l}(x_1, 0) + \tilde{M}^c_{12klmn} \overset{1}{u}_{k,l}(x_1, 0) \overset{1}{u}_{m,n}(x_1, 0) = 0, \qquad \overset{2_s}{u}_2(x_1, 0) = 0. \tag{2.0.5}$$

The conditions on the free surface are,

$$\tilde{L}^c_{i2kl} \overset{2_f}{u}_{k,l}(x_1, L) + \tilde{M}^c_{i2klmn} \overset{1}{u}_{k,l}(x_1, L) \overset{1}{u}_{m,n}(x_1, L) = 0, \qquad i = 1, 2. \tag{2.0.6}$$

Finally, the film-substrate interface boundary and continuity conditions are,

$$\begin{gathered} \left[ \tilde{L}^c_{i2kl} \overset{2_s}{u}_{k,l} + \tilde{M}^c_{i2klmn} \overset{1}{u}_{k,l} \overset{1}{u}_{m,n} \right]_{(x_1, L_1)} = \frac{\mu_f}{\mu_s} \left[ \tilde{L}^c_{i2kl} \overset{2_f}{u}_{k,l} + \tilde{M}^c_{i2klmn} \overset{1}{u}_{k,l} \overset{1}{u}_{m,n} \right]_{(x_1, L_1)}, \quad i = 1, 2, \\ \overset{2_s}{u}(x_1, L_1) = \overset{2_f}{u}(x_1, L_1). \end{gathered} \tag{2.0.7}$$

Using the known form of the initial asymmetric bifurcation mode $\mathcal{A}$, $\overset{1}{u}$ from (1.3.23), we may expand the right hand side of (2.0.4). After grouping terms and simplifying, and with the help of a half-angle identity for the case of $i = 2$, this gives,

$$\begin{aligned} - \left( \tilde{M}^c_{1jklmn} \overset{1}{u}_{k,l} \overset{1}{u}_{m,n} \right)_{,j} &= -\varepsilon \sin(2\omega_c x_1) E_1(x_2), \\ - \left( \tilde{M}^c_{2jklmn} \overset{1}{u}_{k,l} \overset{1}{u}_{m,n} \right)_{,j} &= -\varepsilon \cos(2\omega_c x_1) E_2(x_2) + \widetilde{E}'_2(x_2), \end{aligned} \tag{2.0.8}$$

where we have used the major symmetries of $\tilde{M}^c_{ijklmn}$ and its known zero components, which are found in a manner similar to that used to obtain (1.2.8). The constant $\varepsilon$ is $+1$ for $\mathcal{A}$, and $-1$ for $\mathcal{S}$. The expressions for $E_1$, $E_2$, and $\widetilde{E}_2$ are,

$$\begin{aligned}
E_1(x_2) =& -\tilde{M}^c_{111111}\omega_c^3\,(v_1)^2 + 2\tilde{M}^c_{111122}\omega_c^2\,(v_1)\,(v_2)' - \tilde{M}^c_{112222}\omega_c\,\left((v_2)'\right)^2 \\
&+ \tilde{M}^c_{111212}\omega_c\,\left((v_1)'\right)^2 + 2\tilde{M}^c_{111221}\omega_c^2\,(v_1)'\,(v_2) + \tilde{M}^c_{112121}\omega_c^3\,(v_2)^2 \\
&+ \left[\tilde{M}^c_{121112}\omega_c\,(v_1)\,(v_1)' + \tilde{M}^c_{121121}\omega_c^2\,(v_1)\,(v_2) - \tilde{M}^c_{122212}\,(v_1)'\,(v_2)' - \tilde{M}^c_{122221}\omega_c\,(v_2)\,(v_2)'\right]',
\end{aligned}$$

$$\begin{aligned}
E_2(x_2) =& 2\tilde{M}^c_{211112}\omega_c^2\,(v_1)\,(v_1)' + 2\tilde{M}^c_{211121}\omega_c^3\,(v_1)\,(v_2) - 2\tilde{M}^c_{212221}\omega_c^2\,(v_2)\,(v_2)' - \tilde{M}^c_{212212}\omega_c\,(v_1)'\,(v_2)' \\
&+ \left[\frac{1}{2}\tilde{M}^c_{222222}\left((v_2)'\right)^2 - \tilde{M}^c_{221122}\omega_c\,(v_1)\,(v_2)' + \frac{1}{2}\tilde{M}^c_{221111}\omega_c^2\,(v_1)^2\right. \\
&\left. - \frac{1}{2}\tilde{M}^c_{221212}\left((v_1)'\right)^2 - \frac{1}{2}\tilde{M}^c_{222121}\omega_c^2\,(v_2)^2 - \tilde{M}^c_{221221}\omega_c\,(v_1)'\,(v_2)\right]',
\end{aligned}$$

$$\begin{aligned}
\widetilde{E}_2(x_2) =& \frac{1}{2}\left[\tilde{M}^c_{222222}\left((v_2)'\right)^2 - 2\tilde{M}^c_{221122}\omega_c\,(v_1)\,(v_2)' + \tilde{M}^c_{221111}\omega_c^2\,(v_1)^2\right. \\
&\left. + \tilde{M}^c_{221212}\left((v_1)'\right)^2 + \tilde{M}^c_{222121}\omega_c^2\,(v_2)^2 + 2\tilde{M}^c_{221221}\omega_c\,(v_1)'\,(v_2)\right].
\end{aligned} \tag{2.0.9}$$

We notice that the right hand side of the first equation in (2.0.4) is harmonic in $x_1$ with frequency $2\omega_c$ from (2.0.8). Similarly, the right hand side of the second equation in (2.0.4) has a harmonic part in $x_1$ with frequency $2\omega_c$ with an additional $x_2$ dependence of $\widetilde{E}_2$ from (2.0.8). Inspired by the form of the homogeneous solution in (1.3.23), we search for solutions that are harmonic in $x_1$ with frequency $2\omega_c$,

$$\begin{aligned}
\overset{2}{u}_1(x_1,x_2) &= \begin{cases} \overset{2_s}{u}_1 = \sin(2\omega_c x_1) w_1^s(x_2), & x_2 \in [0, L_1) \\ \overset{2_f}{u}_1 = \sin(2\omega_c x_1) w_1^f(x_2), & x_2 \in [L_1, L_2] \end{cases}, \\
\overset{2}{u}_2(x_1,x_2) &= \begin{cases} \overset{2_s}{u}_2 = \cos(2\omega_c x_1) w_2^s(x_2) + \widetilde{w}_2^s(x_2), & x_2 \in [0, L_1) \\ \overset{2_f}{u}_2 = \cos(2\omega_c x_1) w_2^f(x_2) + \widetilde{w}_2^f(x_2), & x_2 \in [L_1, L_2] \end{cases}.
\end{aligned} \tag{2.0.10}$$

By construction, this form of the solution is in $N^\perp$ from the $2\omega_c$ frequency of the $x_1$ dependence. This form of the solution can be plugged into the PDE (2.0.4). After simplification, this gives,

$$\begin{aligned}
-(2\omega_c)^2\tilde{L}^c_{1111}w_1 + \tilde{L}^c_{1212}w_1'' + (2\omega_c)\left(\tilde{L}^c_{1221} + \tilde{L}^c_{1122}\right)w_2' &= -\varepsilon E_1(x_2), \\
-(2\omega_c)^2\tilde{L}^c_{2121}w_2 + \tilde{L}^c_{2222}w_2'' + (2\omega_c)\left(\tilde{L}^c_{2112} + \tilde{L}^c_{2211}\right)w_1' &= -\varepsilon E_2(x_2), \\
\tilde{L}^c_{2222}\widetilde{w}_2'' &= \widetilde{E}_2',
\end{aligned} \tag{2.0.11}$$

which must be satisfied in both the film and substrate domains. Our form of the solution in (2.0.10) can also be plugged into the boundary conditions (2.0.5)–(2.0.7) giving,

$$
\begin{aligned}
\tilde{L}^c_{1212}\left(w_1^s\right)'(0) - (2\omega_c)\tilde{L}^c_{1221} w_2^s(0) &= -\varepsilon F_1^s(0),\\
w_2^s(0) &= 0,\\
\tilde{L}^c_{1212}\left(w_1^f\right)'(L) - (2\omega_c)\tilde{L}^c_{1221} w_2^f(L) &= -\varepsilon F_1^s(L),\\
(2\omega_c)\tilde{L}^c_{2211} w_1^f(L) + \tilde{L}^c_{2222}\left(w_2^f\right)'(L) &= -\varepsilon F_2^f(L),\\
\Big[\tilde{L}^c_{1212}\left(w_1^s\right)'(L_1) - (2\omega_c)\tilde{L}^c_{1221}\, w_2^s(L_1)\Big] &\\
-\frac{\mu_f}{\mu_s}\left[\tilde{L}^c_{1212}\left(w_1^f\right)'(L_1) - (2\omega_c)\tilde{L}^c_{1221} w_2^f(L_1)\right] &= -\varepsilon\left(F_1^s(L_1) - \frac{\mu_f}{\mu_s}F_1^f(L_1)\right),\\
\Big[(2\omega_c)\tilde{L}^c_{2211} w_1^s(L_1) + \tilde{L}^c_{2222}\left(w_2^s\right)'(L_1)\Big] &\\
-\frac{\mu_f}{\mu_s}\left[(2\omega_c)\tilde{L}^c_{2211}\, w_1^f(L_1) + \tilde{L}^c_{2222}\left(w_2^f\right)'(L_1)\right] &= -\varepsilon\left(F_2^s(L_1) - \frac{\mu_f}{\mu_s}F_2^f(L_1)\right),\\
w_1^s(L_1) - w_1^f(L_1) &= 0,\\
w_2^s(L_1) - w_2^f(L_1) &= 0,
\end{aligned}
\tag{2.0.12}
$$

and

$$
\begin{aligned}
\widetilde{w}_2^s(0) &= 0,\\
\tilde{L}^c_{2222}\widetilde{w}_2^{f\prime}(L) &= -\widetilde{E}_2^f(L),\\
\tilde{L}^c_{2222}\widetilde{w}_2^{s\prime}(L_1) - \frac{\mu_f}{\mu_s}\tilde{L}^c_{2222}\widetilde{w}_2^{f\prime}(L_1) &= \frac{\mu_f}{\mu_s}\widetilde{E}_2^f(L_1) - \widetilde{E}_2^s(L_1),\\
\widetilde{w}_2^s(L_1) - \widetilde{w}_2^f(L_1) &= 0,
\end{aligned}
\tag{2.0.13}
$$

where,

$$
\begin{aligned}
F_1^{s|f}(x_2) =& \tilde{M}^c_{121112}\omega_c v_1^{s|f}\left(v_1^{s|f}\right)' + \tilde{M}^c_{121121}\omega_c^2 v_1^{s|f} v_2^{s|f}\\
&- \tilde{M}^c_{122212}\left(v_1^{s|f}\right)'\left(v_2^{s|f}\right)' - \tilde{M}^c_{122221}\omega_c v_2^{s|f}\left(v_2^{s|f}\right)',\\
F_2^{s|f}(x_2) =& \frac{1}{2}\Bigg[\tilde{M}^c_{222222}\left(\left(v_2^{s|f}\right)'\right)^2 - 2\tilde{M}^c_{221122}\omega_c v_1^{s|f}\left(v_2^{s|f}\right)'\tilde{M}^c_{221111}\omega_c^2\left(v_1^{s|f}\right)^2\\
&- \tilde{M}^c_{221212}\left(\left(v_1^{s|f}\right)'\right)^2 - \tilde{M}^c_{222121}\omega_c^2\left(v_2^{s|f}\right)^2 - 2\tilde{M}^c_{221221}\omega_c\left(v_1^{s|f}\right)' v_2^{s|f}\Bigg].
\end{aligned}
\tag{2.0.14}
$$

The first line in (2.0.12) comes from the traction conditions at the bottom edge in (2.0.5). The second line in (2.0.12) and the first line in (2.0.13) come from the Dirichlet condition on the bottom edge in (2.0.5). The third and fourth line in (2.0.12) and the second line in (2.0.13) come from the traction condition on the free surface from (2.0.7). The fifth and sixth line in (2.0.12) and the third line in (2.0.13) from the interface Neumann conditions from (2.0.7). Finally, the seventh and eighth line in (2.0.12) and the fourth line in (2.0.13) come from the continuity conditions at the interface from (2.0.7). The second order system of in-homogeneous ODE's for $w_1(x_2)$ and $w_2(x_2)$ in (2.0.11) can be written as a first order system for the film and substrate domain as,

$$
\left(\mathbf{w}^{s|f}\right)'(x_2) = \mathbf{A}\mathbf{w}^{s|f}(x_2) + \mathbf{g}^{s|f}(x_2),
\tag{2.0.15}
$$

$$
\mathbf{w}^{s|f}(x_2) = \begin{bmatrix} w_1^{s|f} & \left(w_1^{s|f}\right)' & w_2^{s|f} & \left(w_2^{s|f}\right)' \end{bmatrix}^T, \quad \mathbf{g}^{s|f}(x_2) = \begin{bmatrix} 0 & -\varepsilon\frac{E_1^{s|f}(x_2)}{\tilde{L}^c_{1212}} & 0 - \varepsilon\frac{E_2^{s|f}(x_2)}{\tilde{L}^c_{2222}} \end{bmatrix}^T,
$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (2\omega_c)^2 \frac{\tilde{L}^c_{1111}}{\tilde{L}^c_{1212}} & 0 & 0 & (2\omega_c)\frac{\tilde{L}^c_{1122}+\tilde{L}^c_{2112}}{\tilde{L}^c_{1212}} \\ 0 & 0 & 0 & 1 \\ 0 & -(2\omega_c)\frac{\tilde{L}^c_{1122}+\tilde{L}^c_{2112}}{\tilde{L}^c_{2222}} & (2\omega_c)^2\frac{\tilde{L}^c_{2121}}{\tilde{L}^c_{2222}} & 0 \end{bmatrix}.$$

These systems can then be diagonalized. If $\mathbf{A}$ has eigenvalues $r_i$ with corresponding eigenvectors $\varphi_i$, matrix $\mathbf{\Phi}$ can be defined as,

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \end{bmatrix}. \tag{2.0.16}$$

Then, by defining $\mathbf{h}^{s|f}(x_2)$ as,

$$\mathbf{h}^{s|f}(x_2) = \mathbf{\Phi}^{-1}\mathbf{g}^{s|f}(x_2), \tag{2.0.17}$$

the solution of (2.0.15) can be found by considering both a homogeneous part and a particular part. The particular solution can be obtained by using an integrating factor. The homogeneous solution is a linear combination of exponentials, with coefficients determined by the boundary conditions. These are given by

$$\begin{gathered} w_1^{s|f}(x_2) = w_{1p}^{s|f}(x_2) + w_{1h}^{s|f}(x_2), \\ w_{1p}^{s|f}(x_2) = \sum_{i=4}^{4} \Phi_{1i} e^{r_i x_2} \int_0^{x_2} e^{-r_i\tau} h_i^{s|f}(\tau) d\tau, \qquad w_{1h}^{s|f}(x_2) = \sum_{i=1}^{4} \Phi_{1i} C_i^{s|f} e^{r_i x_2}, \end{gathered} \tag{2.0.18}$$

and

$$\begin{gathered} w_2^{s|f}(x_2) = w_{2p}^{s|f}(x_2) + w_{2h}^{s|f}(x_2), \\ w_{2p}^{s|f}(x_2) = \sum_{i=4}^{4} \Phi_{3i} e^{r_i x_2} \int_0^{x_2} e^{-r_i\tau} h_i^{s|f}(\tau) d\tau, \qquad w_{2h}^{s|f}(x_2) = \sum_{i=1}^{4} \Phi_{3i} C_i^{s|f} e^{r_i x_2}. \end{gathered} \tag{2.0.19}$$

The "$p$" and the "$h$" denote the particular and homogenous solutions, respectfully, and the $C_i^{s|f}$'s are integration constants that must be solved for using boundary conditions. Plugging (2.0.18)–(2.0.19) into the boundary conditions of (2.0.12) gives the following linear equations for $C$'s,

$$\begin{aligned} \sum_{i=1}^{4} \left[\Phi_{1i}\tilde{L}^c_{1212} r_i - \Phi_{3i}(2\omega_c)\tilde{L}^c_{1221}\right] C_i^s &= -\varepsilon F_1^s(0), \\ \sum_{i=1}^{4} \Phi_{3i} C_i^s &= 0, \\ \sum_{i=1}^{4} \left[\Phi_{1i}\tilde{L}^c_{1212} r_i - \Phi_{3i}(2\omega_c)\tilde{L}^c_{1221}\right] e^{r_i L} C_i^f &= Q_1, \\ \sum_{i=1}^{4} \left[\Phi_{1i}(2\omega_c)\tilde{L}^c_{2211} + \Phi_{3i} r_i \tilde{L}^c_{2222}\right] e^{r_i L} C_i^f &= Q_2, \\ \sum_{i=1}^{4} \left[\Phi_{1i}\tilde{L}^c_{1212} r_i - \Phi_{3i}(2\omega_c)\tilde{L}^c_{1221}\right] e^{r_i L_1} C_i^s - \frac{\mu_f}{\mu_s}\sum_{i=1}^{4} \left[\Phi_{1i}\tilde{L}^c_{1212} r_i - \Phi_{3i}(2\omega_c)\tilde{L}^c_{1221}\right] e^{r_i L_1} C_i^f &= Q_3, \\ \sum_{i=1}^{4} \left[\Phi_{1i}(2\omega_c)\tilde{L}^c_{2211} + \Phi_{3i} r_i \tilde{L}^c_{2222}\right] e^{r_i L_1} C_i^s + \frac{\mu_f}{\mu_s}\sum_{i=1}^{4} \left[\Phi_{1i}(2\omega_c)\tilde{L}^c_{2211} + \Phi_{3i} r_i \tilde{L}^c_{2222}\right] e^{r_i L_1} C_i^f &= Q_4, \\ \sum_{i=1}^{4} \Phi_{1i} e^{r_i L_1} C_i^s - \sum_{i=1}^{4} \Phi_{1i} e^{r_i L_1} C_i^f &= w_{1p}^f(L_1) - w_{1p}^s(L_1), \\ \sum_{i=1}^{4} \Phi_{3i} e^{r_i L_1} C_i^s - \sum_{i=1}^{4} \Phi_{3i} e^{r_i L_1} C_i^f &= w_{2p}^f(L_1) - w_{2p}^s(L_1), \end{aligned} \tag{2.0.20}$$

where,

$$
\begin{aligned}
Q_1 =& -\varepsilon F_1^f(L) - \tilde{L}^c_{1212}\left(w^f_{1p}\right)'(L_2) + (2\omega_c)\tilde{L}^c_{1221}w^f_{2p}(L), \\
\\
Q_2 =& -\varepsilon F_2^f(L_2) - (2\omega_c)\tilde{L}^c_{2211}w^f_{1p}(L_2) - \tilde{L}^c_{2222}\left(w^f_{2p}\right)'(L), \\
\\
Q_3 =& -\varepsilon\left(F_1^s(L_1) - \frac{\mu_f}{\mu_s}F_1^f(L_1)\right) - \tilde{L}^c_{1212}\left[\left(w^s_{1p}\right)'(L_1) - \frac{\mu_f}{\mu_s}w^{f\prime}_{1p}(L_1)\right] \\
&+ (2\omega_c)\tilde{L}^c_{1221}\left[\overset{1}{w}_{2p}(L_1) - \frac{\mu_f}{\mu_s}\overset{2}{w}_{2p}(L_1)\right], \\
\\
Q_4 =& -\varepsilon\left(F_2^s(L_1) - \frac{\mu_f}{\mu_s}F_2^f(L_1)\right) - \tilde{L}^c_{2222}\left[\left(w^s_{2p}\right)'(L_1) - \frac{\mu_f}{\mu_s}w^{f\prime}_{2p}(L_1)\right], \\
&+ (2\omega_c)\tilde{L}^c_{2211}\left[w^s_{1p}(L_1) - \frac{\mu_f}{\mu_s}w^f_{1p}(L_1)\right].
\end{aligned}
\tag{2.0.21}
$$

This gives an inhomogenous system of eight linear equations for the eight $C$'s, and can be solved numerically, giving the solutions of $w_1(x_2)$ and $w_2(x_2)$. As for $\widetilde{w}_2(x_2)$, the solution is trivially found to be,

$$
\begin{aligned}
\widetilde{w}^s_2(x_2) &= -\frac{1}{\tilde{L}^c_{2222}}\int_0^{x_2}\widetilde{E}^s_2(\tau)d\tau, \\
\\
\widetilde{w}^f_2(x_2) &= -\frac{1}{\tilde{L}^c_{2222}}\int_{L_1}^{x_2}\widetilde{E}^f_2(\tau)d\tau + \widetilde{w}^s_2(L_1).
\end{aligned}
\tag{2.0.22}
$$

This gives the entire solution for $\overset{2}{u}$.

# 3 Exponential Graded Material

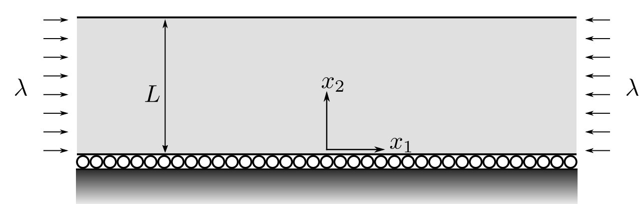


Figure 2: The geometry we will consider for the exponentially graded material, with infinite length in the $x_1$ direction and a finite thickness $L$ in the $x_2$ direction. $\lambda$ is the prescribed average compression.

We consider the compression of a similar 2D strip of material, only now the modulus varies exponentially with depth, as shown in Figure 2. Again, we adopt a plane strain, compressible Neo-Hookean constitutive law,

$$
W(F) = \mu(x_2)\left[\frac{1}{2}\left(I_1 - 2 - \log I_2\right) + \frac{\nu}{1-\nu}\left(\sqrt{I_2} - 1\right)^2\right] = \mu(x_2)\,\tilde{W}(F). \tag{3.0.1}
$$

The parameter $\kappa$ controls the exponential growth through,

$$
\mu(x_2) = \mu_0 e^{\kappa x_2} \tag{3.0.2}
$$

The same treatment that we applied to the film-substrate problem may be used here. First, the principle solution is derived, and is found to be identical to that of the film-substrate problem. Next, we analyze the first bifurcation by solving (1.3.2) for the instability mode $\overset{1}{u}$ and the corresponding critical compression. As expected, $\overset{1}{u}$ takes the form of a surface wave. An identical power series expansion for the bifurcation amplitude and loading parameter is considered, resulting in the same equation for the asymptotic mode $\overset{2}{u}$ as (2.0.1). We then solve the resulting inhomogeneous PDE in a similar manner. While there are slight differences in the derivations for the exponentially graded case, we consider these minor as they follow trivially from the details outlined for the film-substrate problem. Thus, for brevity, we choose to not detail them here.

To analyze the stability of the surface wave bifurcation, a MATLAB® script was developed to compute the asymptotic expansion parameters. Figure 3 shows how these expansion parameters vary with the exponential growth parameter $\kappa$. We see that for $\kappa >\ 2$, the system is stable in displacement control. For $\kappa >\ 7.5$, the system is stable in load control. Thus, for $\kappa < 2$ the system behaves similar to a homogeneous material with unstable surface waves. However, for $\kappa > 7.5$ these surface waves are stabilized in both the displacement and load control settings, and the system behaves similar to the stiff film-substrate. In limited cases, it was found that the stability in load control was dependent on the Poisson ratio. A further study would be required to fully characterize this system.

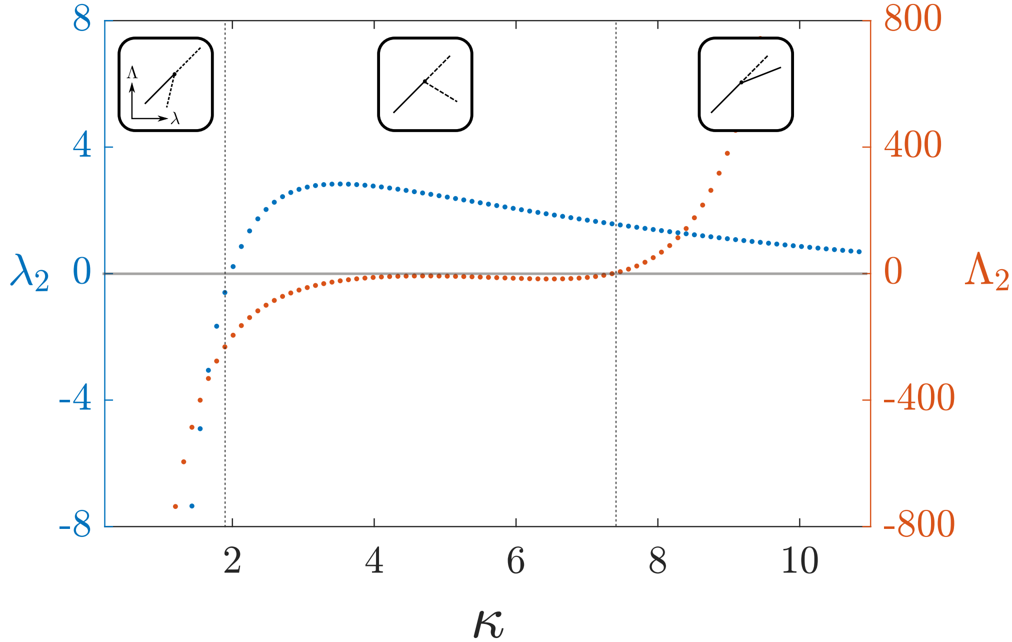


Figure 3: Plot of the asymptotic parameters for varying exponential growth parameter for $\nu = 0.60$. We also illustrate the behavior of the bifurcation diagram for the regions of parameters, that being the $\Lambda$ vs $\lambda$ plot along equilibrium paths.