Supplementary material for:
Active beating modes of two clamped filaments driven by molecular motors

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Analytical expressions for filament shapes

In the following we give the analytical expressions for the integrals that appear in the planar solutions for the two-filament shapes (10-13) in the main text, along with (9). We will refer to incomplete elliptic integrals of the first and second kind according to the following definition [1]

$$F(\phi \mid m) = \int_0^\phi \frac{\mathrm{d}\alpha}{\sqrt{1 - m\sin^2\alpha}} \qquad E(\phi \mid m) = \int_0^\phi \sqrt{1 - m\sin^2\alpha}\,\mathrm{d}\alpha \tag{S1}$$

and for the complete elliptic integrals

$$K(m) = F\left(\frac{\pi}{2} \mid m\right) \qquad E(m) = E\left(\frac{\pi}{2} \mid m\right). \tag{S2}$$

For the sum of two integrals with identical integrands, we use the shortcut

$$\int_a^b f(x)\mathrm{d}x + \int_c^d f(x)\mathrm{d}x = \left[\int_a^b + \int_c^d\right] f(x)\mathrm{d}x. \tag{S3}$$

A. Solution (i)

Solution (i) (shown in Fig. 2a in the main text) is characterised by an inflection point in filament 1. The shapes are determined by equations (10,11). The angle of filament 1 runs from 0 to a maximum ϕ_M and back to the end value ϕ_E. The angle of filament 2 runs monotonously from 0 to ϕ_E. The shape of filament 1 is determined by the solution

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of Eq. (10)

$$\sqrt{\frac{F}{2EI}}L_1 = \left[\int_0^{\phi_M} + \int_{\phi_E}^{\phi_M}\right] \frac{\mathrm{d}\phi}{\sqrt{\cos\phi - \cos\phi_M}}$$
$$= 2\sqrt{2}K\left(\sin^2\frac{\phi_M}{2}\right) - \sqrt{2}F\left(\gamma(\phi_E) \mid \sin^2\frac{\phi_M}{2}\right) \tag{S4}$$

$$\sqrt{\frac{2F}{EI}}x_1(\phi_1) = \left[\int_0^{\phi_M} \mp \int_{\phi_1}^{\phi_M}\right] \frac{\cos\phi}{\sqrt{\cos\phi - \cos\phi_M}}\mathrm{d}\phi$$
$$= \begin{cases} 2\sqrt{2}E\left(\gamma(\phi_1) \mid \sin^2\frac{\phi_M}{2}\right) - \sqrt{2}F\left(\gamma(\phi_1) \mid \sin^2\frac{\phi_M}{2}\right) \\ -2\sqrt{2}K\left(\sin^2\frac{\phi_M}{2}\right) + 4\sqrt{2}E\left(\sin^2\frac{\phi_M}{2}\right) - 2\sqrt{2}E\left(\gamma(\phi_1) \mid \sin^2\frac{\phi_M}{2}\right) + \sqrt{2}F\left(\gamma(\phi_1) \mid \sin^2\frac{\phi_M}{2}\right) \end{cases} \tag{S5}$$

$$\sqrt{\frac{2F}{EI}}z_1(\phi_1) = \left[\int_0^{\phi_M} \mp \int_{\phi_1}^{\phi_M}\right] \frac{\sin\phi}{\sqrt{\cos\phi - \cos\phi_M}}\mathrm{d}\phi = 2\sqrt{1-\cos\phi_M} \mp 2\sqrt{\cos\phi_1 - \cos\phi_M} \tag{S6}$$

with

$$\gamma(\phi) = \arcsin\left(\frac{\sin(\phi/2)}{\sin(\phi_M/2)}\right) \tag{S7}$$

and that of filament 2 by the solution of Eq. (11)

$$\sqrt{\frac{F}{2EI}}L_2 = \int_0^{\phi_E} \frac{\mathrm{d}\phi}{\sqrt{2\cos\phi_E - \cos\phi_M - \cos\phi}} = \frac{2}{\sqrt{2\cos\phi_E - \cos\phi_M - 1}}F\left(\frac{\phi_E}{2} \mid s\right) \tag{S8}$$

$$\sqrt{\frac{2F}{EI}}x_2(\phi_2) = \int_0^{\phi_2} \frac{\cos\phi}{\sqrt{2\cos\phi_E - \cos\phi_M - \cos\phi}}\mathrm{d}\phi$$
$$= -2\sqrt{2\cos\phi_E - \cos\phi_M - 1}\,E\left(\frac{\phi_2}{2} \mid s\right) + 2\frac{2\cos\phi_E - \cos\phi_M}{\sqrt{2\cos\phi_E - \cos\phi_M - 1}}F\left(\frac{\phi_2}{2} \mid s\right) \tag{S9}$$

$$\sqrt{\frac{2F}{EI}}z_2(\phi_2) = \int_0^{\phi_2} \frac{\sin\phi}{\sqrt{2\cos\phi_E - \cos\phi_M - \cos\phi}}\mathrm{d}\phi$$
$$= 2\sqrt{2\cos\phi_E - \cos\phi_M - \cos\phi_2} - 2\sqrt{2\cos\phi_E - \cos\phi_M - 1} \tag{S10}$$

with

$$s = -\frac{2}{2\cos\phi_E - \cos\phi_M - 1}. \tag{S11}$$

B. Solution (ii)

Solution (ii) (Fig. 2b in the main text), described by equations (12,13), has a loop in filament 1. The angle of filament 1 decreases monotonously from π to ϕ_E while the angle of filament 2 increases monotonously from 0 to ϕ_E. The shape of filament 1 is determined by the solution of Eq. (12)

$$\sqrt{\frac{F}{2EI}}L_1 = \int_\pi^{\phi_E} \frac{-\mathrm{d}\phi}{\sqrt{C - \cos\phi}} = \frac{2}{\sqrt{1+C}}F\left(\frac{\pi - \phi_E}{2} \mid r\right) \tag{S12}$$

$$\sqrt{\frac{2F}{EI}}x_1(\phi_1) = \int_\pi^{\phi_1} \frac{-\cos\phi}{\sqrt{C - \cos\phi}}\mathrm{d}\phi$$
$$= -2\sqrt{1+C}E\left(\frac{\pi - \phi_1}{2} \mid r\right) + \frac{2C}{\sqrt{1+C}}F\left(\frac{\pi - \phi_1}{2} \mid r\right) \tag{S13}$$

$$\sqrt{\frac{2F}{EI}}z_1(\phi_1) = \int_\pi^{\phi_1} \frac{-\sin\phi}{\sqrt{C - \cos\phi}}\mathrm{d}\phi = 2\sqrt{1+C} - 2\sqrt{C - \cos\phi_1} \tag{S14}$$

with

$$r = \frac{2}{1+C} \tag{S15}$$

and the shape of filament 2 by the solution of Eq. (13)

$$\begin{aligned}
\sqrt{\frac{F}{2EI}} L_2 &= \int_0^{\phi_E} \frac{\mathrm{d}\phi}{\sqrt{C + \cos\phi - 2\cos\phi_E}} \\
&= \frac{2}{\sqrt{1 + C - 2\cos\phi_E}} F\left(\frac{\phi_E}{2} \mid s\right) \qquad \text{(S16)} \\
\sqrt{\frac{2F}{EI}} x_2(\phi_2) &= \int_0^{\phi_2} \frac{\cos\phi}{\sqrt{C + \cos\phi - 2\cos\phi_E}} \mathrm{d}\phi \\
&= 2\sqrt{1 + C - 2\cos\phi_E} E\left(\frac{\phi_2}{2} \mid s\right) - 2\frac{C - 2\cos\phi_E}{\sqrt{1 + C - 2\cos\phi_E}} F\left(\frac{\phi_2}{2} \mid s\right) \qquad \text{(S17)} \\
\sqrt{\frac{2F}{EI}} z_2(\phi_2) &= \int_0^{\phi_2} \frac{\sin\phi}{\sqrt{C + \cos\phi - 2\cos\phi_E}} \mathrm{d}\phi \\
&= 2\sqrt{1 + C - 2\cos\phi_E} - 2\sqrt{C + \cos\phi_2 - 2\cos\phi_E} \qquad \text{(S18)}
\end{aligned}$$

with

$$s = \frac{2}{1 + C - 2\cos\phi_E}. \tag{S19}$$

[1] I. S. Gradshteyn and I. M. *Table of integrals, series, and products.* Academic Press, 2014.