7. Validation of functional equations through the estimation of characteristic impedances in half-space planar regions

In this section, we further validate the functional equations (5.11)–(5.13) and (5.18)–(5.20) obtained in the general case of three-dimensional angular region problems by computing the characteristic impedances of the half spaces identified as region 1 ($y > 0$) and region 2 ($y < 0$) in figure 3 for planar problems.

Figure 3 shows the half-plane problem (crack) where arbitrary boundary conditions can be applied. We recall that GWHEs for practical problems can be derived from (5.11)–(5.13) and (5.18)–(5.20) by applying specific boundary conditions (traction-free, clamped, etc.). For example, this method can be used to compare with solutions reported in [35,36] for the half-plane problem. In this case, we note that, starting from the general functional equations, by imposing $\gamma = \pi$, we model the half-plane problem via GWHEs that reduce to CWHEs due to the definitions of spectral variables m.

Let us start from region 1, considering (5.11)–(5.13). To model the planar problem, we impose $\gamma = \pi$, $\alpha_o = 0$ and all the continuous z components of the field $\mathbf{T}$ and $\mathbf{v}$ null: $T_{yz} = T_{YZ} = 0$, $v_z = v_Z = 0$. From (5.11)–(5.12) ((5.13) is trivially null in this case) we have

$$\begin{aligned} &Z_o((2\eta^2 - {k_s}^2)v_y + 2\eta v_x \xi_p) + k_s(\eta T_{xy} - T_{yy}\xi_p) \\ &\quad = Z_o((2\eta^2 - {k_s}^2)v_Y + 2\eta v_X \xi_p) - k_s(\eta T_{XY} - T_{YY}\xi_p), \\ &Z_o(({k_s}^2 - 2\eta^2)v_x + 2\eta v_y \xi_s) + k_s(T_{xy}\xi_s + \eta T_{yy}) \\ &\quad = Z_o(({k_s}^2 - 2\eta^2)v_X + 2\eta v_Y \xi_s) - k_s(T_{XY}\xi_s + \eta T_{YY}). \end{aligned} \tag{7.1}$$

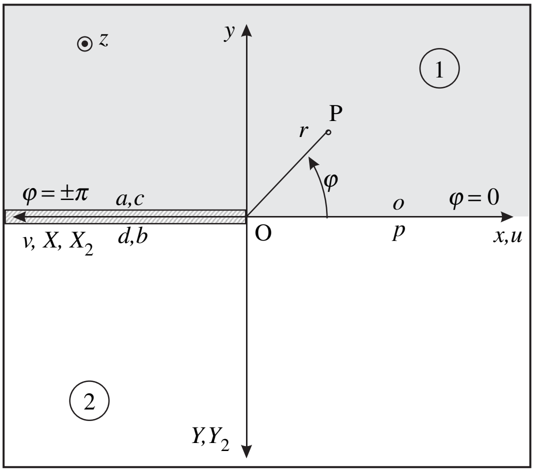

Figure 3. Half-plane planar crack problem with the reference coordinate systems and boundaries adapted from the general configuration reported in figure 2 ($X \equiv X_2$, $Y \equiv Y_2$ local face Cartesian coordinates are reported and are equal in this case due to rotation). The half crack is localized at $x < 0, y = 0$ and the surrounding space is divided into two rectangular regions: region 1 ($y > 0$) and region 2 ($y < 0$). In this section, we evaluate the characteristic impedances of the half-space regions 1 and 2 that are independent from the boundary conditions on the half-plane and implicitly assume absence of sources localized at finite.

Now let us focus attention on the non-null continuous field component of $\mathbf{T}$ and $\mathbf{v}$, we have, respectively, for (2.16) with (4.1) and (5.9) with (4.15)

$$\boldsymbol{\psi}_t = (T_{yy}, T_{xy}, v_x, v_y)' \quad \text{and} \quad \boldsymbol{\psi}_{as} = (-T_{YY}, -T_{XY}, v_X, v_Y)'. \tag{7.2}$$

From the definitions of $\boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_{as}$, respectively, defined in $x > 0, y = 0$ in x, y coordinates and in $x < 0, y = 0_+$ in X, Y coordinates, we estimate the total fields for $y = 0_+$ as

$$\boldsymbol{\psi}_{0+}^{\text{tot}} = \boldsymbol{\psi}_t - \boldsymbol{\psi}_{as} = (T_{yy}^{\text{tot}}, T_{xy}^{\text{tot}}, v_x^{\text{tot}}, v_y^{\text{tot}})'. \tag{7.3}$$

In fact, we note that the local-to-face-a X, Y coordinates have opposite direction with respect to x, y thus the velocity vectors are measured with opposite directions while the tensorial stress components have the same directions because of the double inversion.

With the definition of total fields at $y = 0_+$ (7.3), from (7.1), we derive expressions of T_{yy}^{tot}, T_{xy}^{tot} in terms of v_x^{tot}, v_y^{tot} that in matrix form yields the matrix characteristic impedance of region 1

$$\begin{pmatrix} T_{yy}^{\text{tot}} \\ T_{xy}^{\text{tot}} \end{pmatrix} = \mathbb{Z}_c^+ \begin{pmatrix} v_x^{\text{tot}} \\ v_y^{\text{tot}} \end{pmatrix}, \quad \mathbb{Z}_c^+ = \begin{pmatrix} \frac{\eta Z_o}{\mathrm{k_s}}\left(2 - \frac{\mathrm{ks}^2}{\eta^2+\xi_p\xi_s}\right) & -\frac{\mathrm{k_s} Z_o \xi_s}{\eta^2+\xi_p\xi_s} \\ -\frac{\mathrm{k_s} Z_o \xi_p}{\eta^2+\xi_p\xi_s} & \frac{\eta Z_o}{\mathrm{k_s}}\left(\frac{\mathrm{ks}^2}{\eta^2+\xi_p\xi_s} - 2\right) \end{pmatrix}. \tag{7.4}$$

Note that the definition of the characteristic impedance is independent from boundary conditions on the half-plane and implicitly assumes absence of sources localized at finite. The impedance (7.4) is validated with the admittance $\mathbb{Y}_c^+ = (\mathbb{Z}_c^+)^{-1}$ reported in (2.12.5)–(2.12.8) of [4] where, by mistake, a coefficient 2 is missing in (2.12.7) and (2.12.8). We note that while in §2.12 of [4] the characteristic impedance is evaluated from the homogeneous solution of transverse equations in Fourier domain, in the present work, we have used Laplace transforms with boundary conditions that result in a completely different and independent proof.

Now, let us consider region 2 (figure 3) and the related functional equations (5.18)–(5.20) and (5.17) with (4.22). To model the planar problem, we impose $\gamma = \pi$, $\alpha_o = 0$ and all the continuous z components of the field $\mathbf{T}$ and $\mathbf{v}$ null: $T_{yz} = T_{YZ} = 0$, $v_z = v_Z = 0$. From (5.19)–(5.19) ((5.20) is trivially null in this case) we have

$$\left.\begin{aligned}
&Z_o(v_y(k_s{}^2-2\eta^2)+2\eta v_x\xi_p)-k_s(T_{yy}\xi_p+\eta T_{xy})\\
&\qquad =Z_o(v_Y(k_s{}^2-2\eta^2)+2\eta v_X\xi_p)+k_s(T_{YY}\xi_p+\eta T_{XY}),\\
&Z_o(v_x(2\eta^2-k_s{}^2)+2\eta v_y\xi_s)+k_s(T_{xy}\xi_s-\eta T_{yy})\\
&\qquad =Z_o(v_X(2\eta^2-k_s{}^2)+2\eta v_Y\xi_s)-k_s(T_{XY}\xi_s-\eta T_{YY}).
\end{aligned}\right\} \tag{7.5}$$

Now let us focus attention on the non-null continuous field component of **T** and **v**, we have, respectively, for (2.16) with (4.1) and (5.17) with (4.22)

$$\boldsymbol{\psi}_t=(T_{yy},T_{xy},v_x,v_y)' \quad \text{and} \quad \boldsymbol{\psi}_{bs}=(T_{YY},T_{XY},-v_X,-v_Y)'. \tag{7.6}$$

From the definitions of $\boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_{bs}$, respectively, defined in $x>0, y=0$ in x, y coordinates and in $x<0, y=0_-$ in X, Y coordinates, we estimate the total fields for $y=0_-$ as

$$\boldsymbol{\psi}_{0-}^{\text{tot}}=\boldsymbol{\psi}_t+\boldsymbol{\psi}_{bs}=(T_{yy}^{\text{tot}},T_{xy}^{\text{tot}},v_x^{\text{tot}},v_y^{\text{tot}})'. \tag{7.7}$$

Due to the expressions (7.6), the total field in region 2 (7.7) shows a different sign with respect to the expression of region 1 (7.3) to maintain the same physical meaning. With the definition of total fields at $y=0_-$ (7.7), from (7.5) we derive expressions of T_{yy}^{tot}, T_{xy}^{tot} in terms of v_x^{tot}, v_y^{tot} that in matrix form yield the matrix characteristic impedance of region 2

$$\begin{pmatrix} T_{yy}^{\text{tot}} \\ T_{xy}^{\text{tot}} \end{pmatrix}=\mathbb{Z}_c^-\begin{pmatrix} -v_x^{\text{tot}} \\ -v_y^{\text{tot}} \end{pmatrix}, \quad \mathbb{Z}_c^-=\begin{pmatrix} \frac{\eta Z_o}{k_s}\left(\frac{k_s{}^2}{\eta^2+\xi_p\xi_s}-2\right) & -\frac{k_s Z_o\xi_s}{\eta^2+\xi_p\xi_s} \\ -\frac{k_s Z_o\xi_p}{\eta^2+\xi_p\xi_s} & \frac{\eta Z_o}{k_s}\left(2-\frac{k_s{}^2}{\eta^2+\xi_p\xi_s}\right) \end{pmatrix}. \tag{7.8}$$

The impedance (7.8) is validated with the admittance $\mathbb{Y}_c^-=(\mathbb{Z}_c^-)^{-1}$ reported in §12 at (2.12.5)–(2.12.8) of [4] as discussed for region 1. Note that in (7.8), we have assumed a different sign in the velocity with respect to (7.4) of region 1 due to the different direction of propagation in the two regions. Finally, we recall that the method presented in this paper for the calculation of the characteristic impedances is more general and independent from the one reported in [4].

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