Knot polynomials of open and closed curves - Supplementary Information

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1 A finite form for the bracket polynomial of a polygonal curve with 4 edges

In this section we show that an equivalent finite form of bracket polynomial exists, reducing the computation of the integral to a computation of a few dot and cross products between vectors and some arcsin evaluations. Here we provide a finite form of the bracket polynomial for a polygonal curve of 4 edges. This could lead to the creation of its finite form for more edges.

1.1 Closed curves

The first non-trivial bracket polynomial of a closed polygonal curve is that of a polygon of 4 edges, since a polygon of 3 edges is a triangle in 3-space and all projections give a diagram of no crossings except a set of measure zero which corresponds to irregular projections. Let P_4 denote a polygon of 4 edges, e_1, e_2, e_3, e_4 that connect the vertices $(0,1),(1,2),(2,3)$ and $(3,0)$, respectively. Let $\epsilon_{i,j}$ denote the sign of the crossing between the projections of the edges e_i, e_j when they cross. Notice that $\epsilon_{i,j}$ is independent of the projection direction and can take the values 1 and -1.

Proposition 1.1. *The bracket polynomial of a polygon of 4 edges, e_1, e_2, e_3, e_4, in 3-space, P_4, is equal to:*

$$\langle P_4\rangle = 2|L(e_1,e_3)|(-A^{3\epsilon_{1,3}}) + 2|L(e_2,e_4)|(-A^{3\epsilon_{2,4}}) + (1 - ACN(P_4)) \tag{1}$$

where L denotes the Gauss linking integral and ACN denotes the average crossing number.

Proof. In any projection direction there are 3 possible diagrams that may occur as a projection of P_4: a diagram with no crossing, or a crossing between the projections of e_1, e_3 or a crossing between the projections of e_2, e_4. Notice that not both crossings at the same diagram are possible (the line defined by the projection of e_1 cuts the plane in two regions. Since the projection of e_3 intersects the projection of e_1, the projections of the vertices 2 and 3 lie in different regions. Since e_2 joins vertex 1 with 2 and e_4 joins vertex 3 with 0, e_2, e_4 lie in different regions, thus they cannot cross.) In the case where there is no crossing, the bracket polynomial of that projection is equal to 1. When there is a crossing, the bracket polynomial is equal to $-A^{\pm 3}$, where the sign of the exponent is determined

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by the sign of the crossing in the projection. Since the probability of e_2, e_4 crossing is equal to $2|L(e_2, e_4)|$ and the probability of e_1, e_3 crossing is $2|L(e_1, e_3)|$, then the bracket polynomial is

$$\langle P_4 \rangle = 2|L(e_1, e_3)|(-A^{3\epsilon_{1,3}}) + 2|L(e_2, e_4)|(-A^{3\epsilon_{2,4}}) + (1 - ACN(P_4)) \tag{2}$$

where we used the fact that $ACN(P_4) = 2|L(e_1, e_3)| + 2|L(e_2, e_4)|$. Notice that, due to the connectivity of the polygonal curve, $\epsilon_{1,3} = -\epsilon_{2,4}$, thus Eq. 2 could be expressed as

$$\langle P_4 \rangle = 2|L(e_1, e_3)|(-A^{3\epsilon_{1,3}}) + 2|L(e_2, e_4)|(-A^{-3\epsilon_{1,3}}) + (1 - ACN(P_4))$$

□

1.2 Open curves

In the case of a polygonal curve with 3 edges, we denote E_3, the Kauffman bracket polynomial is always trivial, but the writhe of a diagram of a projection of E_3 can be 0 or ± 1, depending on whether e_1, e_3 cross when projected in a direction $\vec{\xi}$.

Proposition 1.2. *Let E_3 denote a polygonal curve of 3 edges, e_1, e_2, e_3 in 3-space, then the bracket polynomial of E_3 is*

$$\langle E_3 \rangle = 2|L(e_1, e_3)|(-A^3)^{\epsilon_{13}} + (1 - 2|L(e_1, e_3)|)$$

where $\epsilon_{1,3}$ is the sign of $L(e_1, e_3)$

Proof. Consider a polygonal curve of 3 edges e_1, e_2, e_3, (E_3). Then in a projection of E_3, $(E_3)_{\vec{\xi}}$, one either sees no crossings, so $\langle (E_3)_{\vec{\xi}} \rangle = 1$, or there is a crossing between e_1 and e_3, in which case $\langle (E_3)_{\vec{\xi}} \rangle = -A^{\epsilon_{1,3}}$, thus

$$\begin{aligned} \langle E_3 \rangle &= p^{(3)}_{k0,0} + p^{(3)}_{k0,\epsilon_{1,3}}(-A^3)^{\epsilon_{1,3}} \\ &= (1 - 2|L(e_1, e_3)|) + 2|L(e_1, e_3)|(-A^3)^{\epsilon_{1,3}} \end{aligned}$$

where $p^{(3)}_{k0,0} = P(K((E_3)_{\vec{\xi}}) = k0, wr((E_3)_{\vec{\xi}}) = 0)$ and $p^{(3)}_{k0,\epsilon_{1,3}} = P(K((E_3)_{\vec{\xi}}) = k0, wr((E_4)_{\vec{\xi}}) = \epsilon_{1,3})$. □

Let E_4 be composed by 4 edges, e_1, e_2, e_3, e_4, connecting the vertices $(0,1), (1,2), (2,3), (3,4)$, respectively.

Let E_4 denote a polygonal curve of 4 edges. Then, by Propositions A.1 and A.2 (in main manuscript), the only non-trivial bracket polynomial is $k2.1$ and the writhe of the diagram is either 2 or -2. All the possible writhe values in a $k0$ (trivial knotoid) diagram of E_4 can be determined by inspection of all the possible diagrams of a polygonal curve of 4 edges, given in Proposition A.1. Let us denote these diagrams as $k0_{A_1}, k0_{A_2}, k0_{A_3}, k0_{B_i}, k0_{B_{i'}}, k0_{B_{ii}}, k0_{B_{ii'}}, k0_C$. Let us denote by wr the writhe of a diagram. Then one can see that $wr(k0_{A_1}) = \pm 1, wr(k0_{A_2}) = \pm 1, wr(k0_{A_3}) = \pm 1, wr(k0_{B_i}) = 0$ or $= \pm 2$, $wr(k0_{B_{i'}}) = 0$ or ± 2, $wr(k0_{B_{ii}}) = 0$ or ± 2, $wr(k0_{B_{ii'}}) = 0$ or ± 2, $wr(k0_C) = \pm 1$. Thus the bracket polynomial of E_4 has the following form:

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$$
\begin{aligned}
\langle E_4\rangle &= p^{(4)}_{k2.1}\langle k2.1\rangle + \sum_{j=-2}^{2} p^{(4)}_{k0,j}(-A^3)^j \\
&= p^{(4)}_{k2.1}(A^2 - A^{-4} + 1) + \sum_{j=-2}^{2} p^{(4)}_{k0,j}(-A^3)^j
\end{aligned}
$$

where $p^{(4)}_{k2.1} = P(K((E_4)_{\vec{\xi}}) = k2.1)$ denotes the geometric probability that a projection of E_4 gives the non-trivial knotoid $k2.1$ (obtained in Theorem A.2 in main manuscript) and where $p^{(4)}_{k0,j} = P(K((E_4)_{\vec{\xi}}) = k0, wr((E_4)_{\vec{\xi}}) = j)$ denotes the probability of obtaining a diagram of the trivial knotoid with writhe j. The rest of this section is focused on obtaining finite forms for these probabilities.

Theorem 1.1. *Let E_4 denote a polygonal curve of 4 edges, e_1, e_2, e_3, e_4 in 3-space, then the bracket polynomial of E_4 is*

$$
\begin{aligned}
\langle E_4\rangle =& p^{(4)}_{k2.1}\langle k2.1\rangle + p^{(4)}_{k0,\epsilon_{2,4}}(-A^3)^{\epsilon_{2,4}} + p^{(4)}_{k0,-\epsilon_{2,4}}(-A^3)^{-\epsilon_{2,4}} + p^{(4)}_{k0,-2\epsilon_{2,4}}(-A^3)^{-2\epsilon_{2,4}} \\
&+ p^{(4)}_{k0,2\epsilon_{2,4}}(-A^3)^{2\epsilon_{2,4}} + p^{(4)}_{k0,0}
\end{aligned}
$$

where the coefficients are:

$$
p^{(4)}_{k2.1} = \begin{cases} \frac{1}{2\pi}A(Q), & \textit{if } \epsilon_{1,3} = \epsilon_{1,4} \\ 0, & \textit{otherwise} \end{cases} \tag{3}
$$

$$
p^{(4)}_{k0,\epsilon_{2,4}} = \begin{cases} 2|L(e_2,e_4)| - \frac{1}{2\pi}A(Q_{4,2,1}), \epsilon_{1,3} = \epsilon_{1,4} \\ 2|L(e_2,e_4)| + 2|L(e_1,e_4)| - \frac{1}{2\pi}(A(Q_{4,2,1}) + A(Q_2) + A(Q_1)), & \textit{if } \epsilon_{2,4} = \epsilon_{1,4} = -\epsilon_{1,3} \\ 2|L(e_2,e_4)| + 2|L(e_1,e_3)| - \frac{1}{2\pi}(A(Q_{4,2,1}) + A(Q_1)), & \textit{if } \epsilon_{2,4} = \epsilon_{1,3} = -\epsilon_{1,4} \end{cases} \tag{4}
$$

$$
p^{(4)}_{k0,-\epsilon_{2,4}} = \begin{cases} 2|L(e_1,e_3)| + 2|L(e_1,e_4)| - \frac{1}{2\pi}(A(Q_{1,3,4}) + A(Q_2) + A(Q_1)), \epsilon_{1,3} = \epsilon_{1,4} \\ 2|L(e_1,e_3)| - \frac{1}{2\pi}A(Q_{1,3,4}), & \textit{if } \epsilon_{2,4} = \epsilon_{1,4} = -\epsilon_{1,3} \\ 2|L(e_1,e_4)| - \frac{1}{2\pi}(A(Q_{1,3,4}) + A(Q_2)), & \textit{if } \epsilon_{2,4} = \epsilon_{1,3} = -\epsilon_{1,4} \end{cases} \tag{5}
$$

$$
p^{(4)}_{k0,2\epsilon_{2,4}} = \begin{cases} \frac{1}{2\pi}(A(Q_2) - A(Q)), & \textit{if } \epsilon_{2,4} = \epsilon_{1,4} = -\epsilon_{1,3} \\ 0, & \textit{otherwise} \end{cases} \tag{6}
$$

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$$p^{(4)}_{k0,-2\epsilon_{2,4}} = \begin{cases} \frac{1}{2\pi}(A(Q_1) - A(Q)), & \textit{if } \epsilon_{1,4} = \epsilon_{1,3} = -\epsilon 2,3 \\ 0, & \textit{otherwise} \end{cases} \tag{7}$$

and

$$p^{(4)}_{k0,0} = 1 - p^{(4)}_{k0,-2\epsilon_{2,4}} + p^{(4)}_{k0,2\epsilon_{2,4}} + p^{(4)}_{k0,-\epsilon_{2,4}} + p^{(4)}_{k0,\epsilon_{2,4}} + p^{(4)}_{k2.1,-\epsilon_{2,4}} \tag{8}$$

where $\epsilon_{i,j}$ denotes the sign of the linking number between e_i, e_j, $Q_1 = Q_{1,3,4} \setminus Q_{2,4}$, $Q_2 = Q_{4,2,1} \setminus Q_{1,3}$ and $Q = Q((E_4)_{\vec{\xi}} = k2.1)$. $P(Q)$ is derived in Theorem A.2 (in main manuscript) and Q_1 is shown in Table 1. $Q_{4,2,1}$, Q_2 are derived with the same formulas for the reversed polygonal curve.

Proof. In the following, for simplicity, we will write $P(A_1)$ to express the probability $P(K((E_4)_{\vec{\xi}}) = k0_{A_1})$, etc.

By Proposition A.2 (in main manuscript), $k2.1$ is a possible knotoid diagram only when $\epsilon_{1,3} = \epsilon_{1,4}$, in which case, it also implies that $\epsilon_{2,4} = -\epsilon_{1,3}$. The probability of obtaining $k2.1$ is found in Theorem A.2 (in main manuscript).

Thus, we only need to examine the probabilities of obtaining the trivial knotoid with a given writhe. By inspection of the diagrams shown in Figure 6 (in main manuscript), we first notice the following:

$$p^{(4)}_{k0,\epsilon_{2,4}} = \begin{cases} P(A_2), & \text{if } \epsilon_{1,3} = \epsilon_{1,4} = -\epsilon_{2,4} \\ P(A_3) + P(A_2) + P(C), & \text{if } \epsilon_{2,4} = \epsilon_{1,4} = -\epsilon_{1,3} \\ P(A_1) + P(A_2) + P(C), & \text{if } \epsilon_{2,4} = \epsilon_{1,3} = -\epsilon_{1,4} \end{cases} \tag{9}$$

$$p^{(4)}_{k0,-\epsilon_{2,4}} = \begin{cases} P(A_1) + P(A_3) + P(C), & \text{if } \epsilon_{1,3} = \epsilon_{1,4} = -\epsilon_{2,4} \\ P(A_1), & \text{if } \epsilon_{2,4} = \epsilon_{1,4} = -\epsilon_{1,3} \\ P(A_3), & \text{if } \epsilon_{2,4} = \epsilon_{1,3} = -\epsilon_{1,4} \end{cases} \tag{10}$$

$$p^{(4)}_{k0,2\epsilon_{2,4}} = \begin{cases} P(B_{ii}) + P(B_{ii'}), & \text{if } \epsilon_{2,4} = \epsilon_{1,4} = -\epsilon_{1,3} \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

$$p^{(4)}_{k0,-2\epsilon_{2,4}} = \begin{cases} P(B_i) + P(B_{i'}), & \text{if } \epsilon_{1,3} = -\epsilon_{1,4} = -\epsilon_{2,4} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

We will compute these probabilities in the three cases: $\epsilon_{1,3} = \epsilon_{1,4}$, $\epsilon_{1,3} = -\epsilon_{1,4} = \epsilon_{2,4}$, $\epsilon_{1,3} = -\epsilon_{1,4} = -\epsilon_{2,4}$.

First, we notice that, in all cases, due to the connectivity of the polygonal curve,

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$\epsilon_{1,3}=\epsilon_{1,4},\ w<0, w_0<0$	$Q_{1,3,4}$	Q_1
$c_{3,1}>0, c_{4,1}>0, c_{3,0}>0, c_{4,0}>0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_2,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3)\cup Q$
$c_{3,1}>0, c_{4,1}>0, c_{3,0}<0, c_{4,0}<0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_2,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1)\cup Q$
$c_{3,1}>0, c_{4,1}>0, c_{3,0}>0, c_{4,0}<0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)\cup Q$
$c_{3,1}>0, c_{4,1}>0, c_{3,0}<0, c_{4,0}>0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)\cup Q)$
$c_{3,1}<0, c_{4,1}<0, c_{3,0}>0, c_{4,0}>0$	$(\vec{n}_4,-\vec{u}_3,-\vec{u}_2,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3)\cup Q$
$c_{3,1}<0, c_{4,1}<0, c_{3,0}<0, c_{4,0}<0$	$(\vec{n}_4,-\vec{u}_3,-\vec{u}_2,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1)\cup Q$
$c_{3,1}<0, c_{4,1}<0, c_{3,0}>0, c_{4,0}<0$	$(\vec{n}_4,-\vec{u}_3,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)\cup Q$
$c_{3,1}<0, c_{4,1}<0, c_{3,0}<0, c_{4,0}>0$	$(\vec{n}_4,-\vec{u}_3,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)\cup Q$
$c_{3,1}>0, c_{4,1}<0, c_{3,0}>0, c_{4,0}>0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_3,-\vec{u}_2,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3)\cup Q$
$c_{3,1}>0, c_{4,1}<0, c_{3,0}<0, c_{4,0}<0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_3,-\vec{u}_2,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1)\cup Q$
$c_{3,1}>0, c_{4,1}<0, c_{3,0}>0, c_{4,0}<0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_3,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)\cup Q$
$c_{3,1}>0, c_{4,1}<0, c_{3,0}<0, c_{4,0}>0$	$(\vec{n}_4,\vec{n}_1,-\vec{u}_3,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)\cup Q$
$c_{3,1}<0, c_{4,1}>0, c_{3,0}>0, c_{4,0}>0$	$(\vec{n}_4,-\vec{u}_3,\vec{n}_1,-\vec{u}_2,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3)\cup Q$
$c_{3,1}<0, c_{4,1}>0, c_{3,0}<0, c_{4,0}<0$	$(\vec{n}_4,-\vec{u}_3,\vec{n}_1,-\vec{u}_2,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1)\cup Q$
$c_{3,1}<0, c_{4,1}>0, c_{3,0}>0, c_{4,0}<0$	$(\vec{n}_4,-\vec{u}_3,\vec{n}_1,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,-\vec{u}_1,\vec{n}_3)\cup Q$
$c_{3,1}<0, c_{4,1}>0, c_{3,0}<0, c_{4,0}>0$	$(\vec{n}_4,-\vec{u}_3,\vec{n}_1,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)$	$(\vec{n}_4,-\vec{v}_3,-\vec{u}_2,\vec{n}_3,-\vec{u}_1)\cup Q$
$\epsilon_{1,3}=\epsilon_{1,4},\ w>0$ or $w_0>0$	$Q_{1,3,4}$	Q_1
	$\emptyset$	$\emptyset$
$\epsilon_{1,3}=-\epsilon_{1,4}, w<0$	$Q_{1,3,4}$	Q_1
$c_{4,0}>0, c_{4,1}>0$	$(\vec{n}_2,-\vec{u}_1,-\vec{u}_2,-\vec{u}_3)$	$Q_{1,3,4}\setminus(\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{n}_2)$
$c_{4,0}<0, c_{4,1}<0$	$(\vec{n}_2,\vec{n}_1,-\vec{u}_2,\vec{n}_3)$	$Q_{1,3,4}$
$c_{4,0}<0, c_{4,1}>0$	$(\vec{n}_2,\vec{n}_1,-\vec{u}_2,-\vec{u}_3)$	$Q_{1,3,4}\setminus(\vec{v}_1,\vec{v}_2,\vec{n}_1,\vec{n}_2)$
$c_{4,0}>0, c_{4,1}<0$	$(\vec{n}_2,-\vec{u}_1,-\vec{u}_2,\vec{n}_3)$	$Q_{1,3,4}$
$\epsilon_{1,3}=-\epsilon_{1,4}, w>0$	$Q_{1,3,4}$	Q_1
$c_{4,0}>0, c_{4,1}>0$	$(\vec{n}_2,-\vec{u}_1,\vec{n}_4,-\vec{u}_3)$	$Q_{1,3,4}\setminus(-\vec{u}_3,\vec{n}_4,\vec{v}_3,\vec{n}_2)$
$c_{4,0}<0, c_{4,1}<0, c_{4',1'}>0$	$(\vec{n}_2,\vec{n}_1,\vec{n}_4,\vec{n}_3)$	$Q_{1,3,4}\setminus(\vec{v}_3,-\vec{v}_2,\vec{n}_2,\vec{n}_1,\vec{n}_4)$
$c_{4,0}<0, c_{4,1}<0, c_{4',1'}<0$	$(\vec{n}_2,\vec{n}_1,\vec{n}_4,\vec{n}_3)$	$Q_{1,3,4}\setminus(\vec{v}_3,-\vec{v}_2,\vec{n}_1,\vec{n}_4)$
$c_{4,0}<0, c_{4,1}>0$	$(\vec{n}_2,\vec{n}_1,\vec{n}_4,-\vec{u}_3)$	$\emptyset$
$c_{4,0}>0, c_{4,1}<0$	$(\vec{n}_2,-\vec{u}_1,\vec{n}_4,\vec{n}_3)$	$Q_{1,3,4}$

Table 1: *The spherical polygons $Q_{1,3,4}$ and $Q_1=Q_{1,3,4}\setminus Q_{2,4}$, respectively, are computed by using the above expressions. The expression $(\vec{w}_1,\dots,\vec{w}_n)$ denotes the spherical polygon defined by the intersection of the great circles with normal vectors $\vec{w}_1,\dots,\vec{w}_n$ in the counterclockwise orientation (see Definition A.1 in main manuscript). The expressions depend on the conformation of the curve in 3-space, where $c_{3,1}=(\vec{p}_{3,1}\cdot\vec{n}_1)_{\epsilon_{1,3}}$, $c_{4,1}=(\vec{p}_{4,1}\cdot\vec{n}_1)\epsilon_{1,3}$, $c_{3,0}=(\vec{p}_{3,0}\cdot\vec{n}_3)\epsilon_{1,3}$, $c_{4,0}=(\vec{p}_{4,0}\cdot\vec{n}_3)\epsilon_{1,3}$, $c_{4',1'}=(\vec{p}_{1,4}\cdot(-\vec{v}_2))\epsilon_{2,4}$ and $w=(\vec{u}_2\times(-\vec{n}_2))\cdot(\vec{u}_2\times\vec{n}_4)$, where $\vec{n}_1,\vec{u}_i,\vec{v}_i$ are the normal vectors to the quadrilaterals $T_{1,3},T_{1,4},T_{2,4}$ and where $\vec{p}_{i,j}$ is the vector that connects vertex i to vertex j in 3-space. The areas of $Q_{4,2,1}$ and Q_2 are obtained from the areas $Q_{1,3,4}$ and Q_1 of the polygonal curve with reversed orientation (see proof of Theorem 1.1).*

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$$Q_{2,4} \cap Q_{1,3} \subset Q_{2,4} \cap Q_{1,4} = Q_{4,2,1} \tag{13}$$
$$Q_{2,4} \cap Q_{1,3} \subset Q_1 = Q_{1,3} \cap Q_{1,4} = Q_{1,3,4}$$

The probabilities can be expressed as:

$$P(A_1) = 2|L(e_1, e_3)| - \frac{1}{2\pi} A(Q_{1,3,4})$$
$$P(A_2) = 2|L(e_1, e_3)| - \frac{1}{2\pi} A(Q_{4,2,1})$$
$$P(A_3) = 2|L(e_1, e_4)| - \frac{1}{2\pi} A(Q_{4,2,1} \setminus Q_{1,3}) - A(Q_{1,3,4})$$
$$P(C) = \frac{1}{2\pi} A(Q_{2,4} \cap Q_{1,3,4})$$
$$P(B_i) + P(B_{i'}) = \frac{1}{2\pi} A(Q_{1,3,4} \setminus Q_{2,4})$$
$$P(B_{ii}) + P(B_{ii'}) = \frac{1}{2\pi} A(Q_{4,2,1} \setminus Q_{1,3})$$

From all these equations, and using the notation $Q_1 = Q_{1,3,4} \setminus Q_{2,4}$ and $Q_2 = Q_{4,2,1} \setminus Q_{1,3}$, we obtain the expressions of the statement of the Theorem.

We proceed with finding finite forms for $Q_{1,3,4}$ and Q_1 from which the finite forms of $Q_{4,2,1}$ and Q_2 are also derived.

Finite form of $Q_{1,3,4}$

The finite form of $Q_{1,3,4}$ is found by Theorem A.1 (in main manuscript) for $i = 0, j = 2$.

Finite form of $Q_{4,2,1}$:

For the finite form of $Q_{4,2,1}$ we think as follows: Let $R(E_4)$ to denote the polygonal curve E_4 with reversed numbering of vertices. Let us denote its edges e_1', e_2', e_3', e_4'. Then $Q_{4,2,1} = Q_{1',3',4'}$. This can be obtained from table 1 determined by the algorithm described in Section 2(a)(i) for $n_{i'}, u_{i'}$ which are related to the normal vectors of E_4 as follows: $\vec{n}_{1'} = -\vec{v}_2$, $\vec{n}_{2'} = -\vec{v}_1$, $\vec{n}_{3'} = -\vec{v}_4$, $\vec{n}_{4'} = -\vec{v}_3$, $\vec{u}_{1'} = -\vec{u}_2$, $\vec{u}_{2'} = -\vec{u}_1$, $\vec{u}_{3'} = -\vec{u}_4$, $\vec{u}_{4'} = -\vec{u}_3$. Accordingly, $w' = (\vec{u}_1 \times \vec{v}_1) \cdot (\vec{u}_1 \times (-\vec{v}_3))$, $w_{0'} = (\vec{n}_4 \times \vec{v}_2) \cdot (\vec{n}_4 \times \vec{n}_3)$, $\epsilon_{1',3'} = \epsilon_{2,4}$ and $\epsilon_{1',4'} = \epsilon_{1,4}$. Finally, $c_{3',1'} = (\vec{p}_{3',1'} \cdot \vec{n}_{1'})\epsilon_{1',3'} = (\vec{p}_{1,3} \cdot (-\vec{v}_2)_{\epsilon 2,4}$, otherwise $c_{4',1'} = (\vec{p}_{4',1'} \cdot \vec{n}_{1'})\epsilon_{1',3'} = (\vec{p}_{1,4} \cdot (-\vec{v}_2))\epsilon_{2,4}$, $c_{3',0'} = (\vec{p}_{3',0'} \cdot \vec{n}_{3'})\epsilon_{1',3'} = (\vec{p}_{1,4} \cdot (-\vec{v}_4))\epsilon_{2,4}$, otherwise $c_{4',0'} = (\vec{p}_{4',0'} \cdot \vec{n}_{3'})\epsilon_{1',3'} = (\vec{p}_{0,4} \cdot (-\vec{v}_4))\epsilon_{2,4}$, when $\epsilon_{1',3'} = \epsilon_{1',4'}$ and $c_{4',0'} = (\vec{p}_{4',0'} \cdot \vec{n}_{1'})\epsilon_{1',3'} = (\vec{p}_{0,4} \cdot (-\vec{v}_2))\epsilon_{2,4}$, otherwise $c_{4',1'} = (\vec{p}_{4',1'} \cdot \vec{n}_{3'})\epsilon_{1',3'} = (\vec{p}_{0,3} \cdot (-\vec{v}_4))\epsilon_{2,4}$, when $\epsilon_{1',3'} = -\epsilon_{1',4'}$

Finite form of Q_1

- Case $\epsilon_{1,4} = \epsilon_{1,3} = -\epsilon_{2,4}$: One can derive from the proof of Theorem A.2 (in main manuscript) the area of $Q_{1,3,4} \setminus Q_{2,4}$. The area will be $Q_1 = Q \cup (\vec{n}_4, -\vec{v}_3, -\vec{u}_2, x)$, where x is equal to $-\vec{u}_1$ or $\vec{n}_3$ or $\vec{n}_3, -\vec{u}_1$ or $-\vec{u}_1, \vec{n}_3$, depending on the signs of $c_{0,3}, c_{0,4}$ (see Table 1).

Next, we consider the case $\epsilon_{1,4} = -\epsilon_{1,3}$ and refer to Figure 1 as an illustrative example. Since $\vec{u}_3 = -\vec{v}_1$ and $\vec{n}_3 = \vec{v}_4$, these spherical edges (which bound $Q_{2,4}$) do not cross the interior of $Q_{1,3,4}$. In order to find $Q_1 = Q_{1,3,4} \setminus Q_{2,4}$, we examine if and how $\vec{v}_2$ and $\vec{v}_3$ intersect the interior of $Q_{1,3,4}$. Figure 1 shows the relative positions of $\vec{v}_1, \vec{v}_4, \vec{v}_2$ determined by the connectivity of the polygonal curve and the orientations of $\vec{v}_1, \vec{v}_4$ are also given by the known orientations of $\vec{u}_3$ and $\vec{n}_3$.

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- Case $\epsilon_{1,4} = -\epsilon_{1,3} = \epsilon_{2,4}$: (This is the case where $c_{4,1} < 0$ in Table 1). This corresponds to the case where $\epsilon_{1\prime,4\prime} = \epsilon_{1\prime,3\prime}$ for the reversed walk. First of all, in this case, we notice that when $c_{4,0} > 0$, then $w\prime > 0$ and, similarly, when $w < 0$ then $w_0\prime > 0$, thus in these cases $Q_{4,2,1} = \emptyset$, giving $Q_1 = Q_{1,3,4}$. Thus, the only case that might give $Q_{2,4} \cap Q_{1,3,4} \neq \emptyset$ is the case $w > 0, c_{4,0} < 0$, equivalently, $w > 0, w_0 < 0$, (see Figure 1). In that case the great circle with normal vector $\vec{v}_3$ intersects the interior of $Q_{1,3,4}$ (since the face with normal vector $\vec{v}_3$ is in-between the faces with normal vectors $\vec{n}_1, \vec{n}_3$). To examine the intersection of $Q_{2,4} \cap Q_{1,3,4}$, we examine the reversed oriented polygon, $R(E_4)$ (see previous paragraph). The above conditions correspond to the case where $\epsilon_{1\prime,3\prime} = \epsilon_{1\prime,4\prime}$, $w\prime < 0, w_0\prime < 0$, which is the case that can give the non-trivial knotoid. Thus, using Theorem **??**, we derive that for $w\prime < 0$, if $c_{4\prime,1\prime} > 0$, then $Q_1 = Q_{1,3,4} \setminus (v_3, -v_2, n_2, n_1, n_4)$ and if $c_{4\prime,1\prime} < 0$, then $Q_1 = Q_{1,3,4} \setminus (v_3, -v_2, n_1, n_4)$.
- Case $\epsilon_{1,4} = -\epsilon_{1,3} = -\epsilon_{2,4}$: (This is the case where $c_{4,1} > 0$ in Table 1) As in the previous case, in order to find $Q_1 = Q_{1,3,4} \setminus Q_{2,4}$, we need the area of $Q_{1,3,4}$ that is determined by the great circles $\vec{v}_2$ and $\vec{v}_3$. To find these intersections, we will examine $Q_{4,2,1}$ using the reverse walk with $\epsilon_{1\prime,3\prime} = -\epsilon_{1\prime,4\prime}$, and we notice that in all cases, $c_{1\prime,4\prime} = (p_{4\prime,1\prime} \cdot \vec{n}_3\prime)\epsilon_{1\prime,3\prime} = (p_{0,3} \cdot (-\vec{v}_4))\epsilon_{2,4} = (p_{0,3} \cdot (-\vec{n}_3))\epsilon_{1,3} > 0$. Indeed, since $\vec{n}_3$ is the normal vector to the face defined by the vertices 1,2,3, of the tetrhedral $T_{1,4}$ and points inwards if $\epsilon_{1,3} > 0$ (in the direction of vertex 3) or outwards otherwise. Thus $c_{1\prime,4\prime} > 0$ in all cases. Thus, the intersection will depend on the sign of $c_{0\prime,4\prime} = (p_{4\prime,0\prime} \cdot \vec{n}_1\prime)\epsilon_{1\prime,3\prime} = (p_{0,4} \cdot (-\vec{v}_2))\epsilon_{2,4}$. This sign will depend on the sign of $c_{4,0} = (\vec{p}_{4,0} \cdot \vec{n}_1)\epsilon_{1,3}$ and the sign of w, which determines if $\vec{u}_2$ lies between $\vec{n}_2, \vec{n}_4$.

If $c_{4,0} < 0$ then $w\prime < 0$ since we can verify that the face with normal vector $\vec{u}_1$ is between the faces with normal vectors $\vec{v}_1, \vec{v}_3$, and $w\prime > 0$ if $c_{4,0} > 0$. If $w < 0$ then $c_{0\prime,4\prime} = (p_{4\prime,0\prime} \cdot \vec{n}_1\prime)\epsilon_{1\prime,3\prime} = (p_{0,4} \cdot (-\vec{v}_2))\epsilon_{2,4} < 0$ since $\vec{v}_2$ points in the opposite direction of the region that contains the vertex 0 when $\epsilon_{2,4} < 0$, and $c_{0\prime,4\prime} > 0$ if $w > 0$.

Thus, by using Table 1 for the reversed walk we find that if $c_{4,0} < 0$ and $w < 0$, then $Q_1 = Q_{1,3,4} \setminus (v_1, v_2, n_1, n_2)$. If $c_{4,0} < 0$ and $w > 0$, then $Q_1 = \emptyset$. If $c_{4,0} > 0$ and $w < 0$, then $Q_1 = Q_{1,3,4} \setminus (v_1, v_2, v_3, n_2)$. If $c_{4,0} > 0$ and $w > 0$, then $Q_1 = Q_{1,3,4} \setminus (v_1, n_4, v_3, n_2)$.

Finite form of Q_2:

For the finite form of Q_2 we think as follows: Let $R(E_4)$ to denote the polygonal curve E_4 with reversed numbering of vertices as described in the Finite form of $Q_{4,2,1}$. Then $Q_2 = Q_{4,2,1} \setminus Q_{1,3} = Q_{1',3',4'} \setminus Q_{2',4'} = Q'_1$, which is found earlier.

□

Example: (continuation of Example in main manuscript) Figure 2 shows the Kauffman bracket polynomial of the open 3-dimensional curve in time and that of the standard diagram of the knotoid k2.1. We see the bracket polynomial of the open curve vary continuously in time, tending to that of the diagram, due to the tightening of the configuration to become almost planar.

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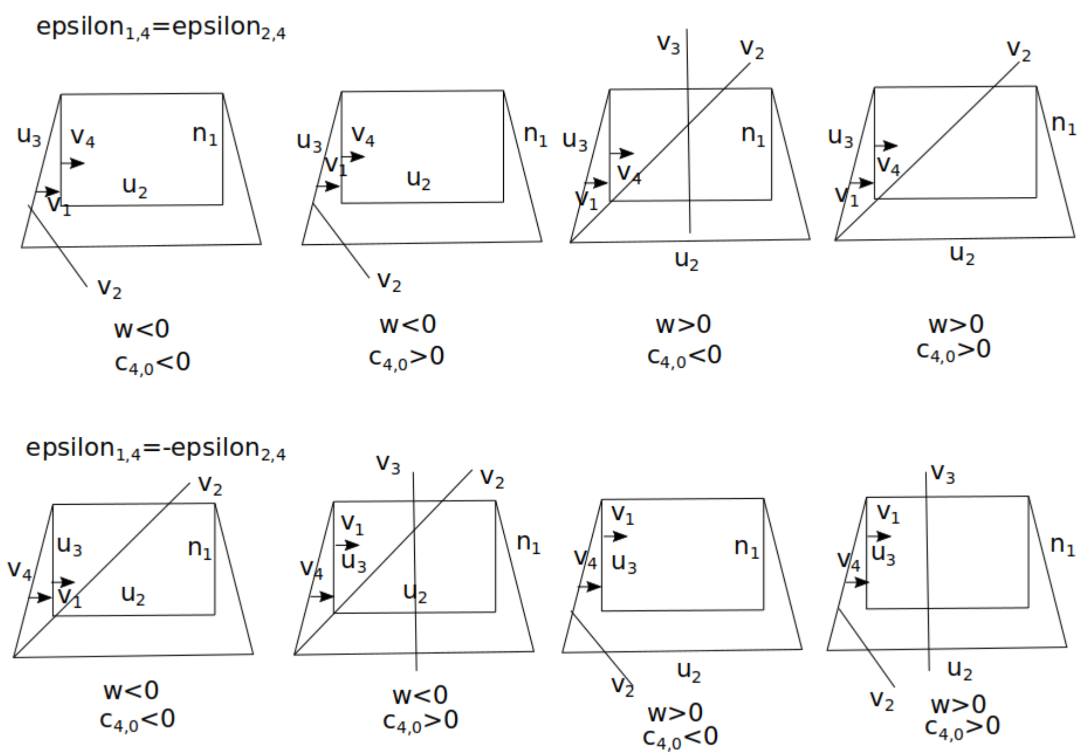

Figure 1: *Representation of $Q_{1,3,4}$ when $\epsilon_{1,3} = \epsilon_{2,4}$. In this case, one great circle of the boundary of $Q_{1,3,4}$ is the one with normal vector $\vec{n}_2$ (top boundary in the figure). The lower great circle (bottom boundary) is $\vec{u}_2$ or $\vec{n}_4$, depending on whether $\epsilon_{1,4} = \epsilon_{2,4}$ or not (equivalently, depending on the sign of $c_{1,4}$). Similar considerations define the other boundaries, where $c_{4,0} = (\vec{p}_{4,0} \cdot \vec{n}_1)\epsilon_{1,3}$, $w = (\vec{u}_2 \times (-\vec{n}_2)) \cdot (\vec{u}_2 \times \vec{n}_4)$, $\vec{n}_3 = \vec{v}_4$ and $\vec{u}_3 = -\vec{v}_1$. To determine $Q_1 = Q_{1,3,4} \setminus Q_{2,4}$, we examine how $\vec{v}_2$ and $\vec{v}_3$ intersect $Q_{1,3,4}$ (see proof of Theorem 1.1). The results are shown in Table 1.*

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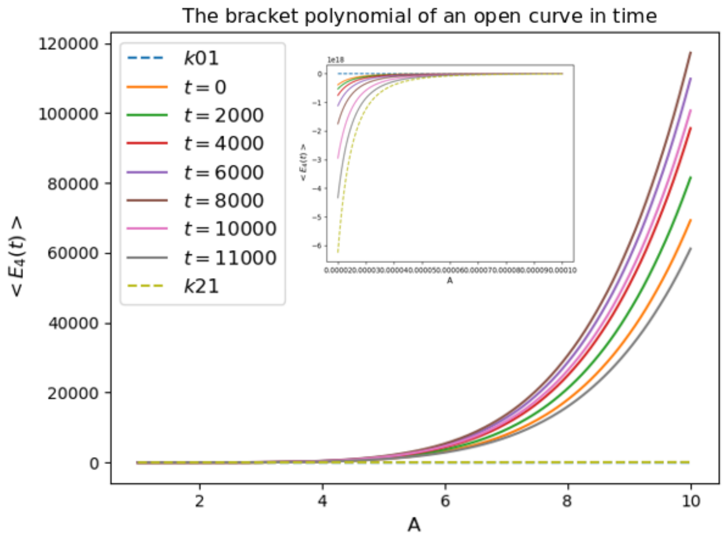

Figure 2: *The Kauffman bracket polynomial of an open polygonal curve as it moves in time. The inset plot shows the polynomial for values of the parameter A less than 1.*

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