SUPPLEMENTRARY INFORMATION: THE GODBILLON-VEY INVARIANT AS TOPOLOGICAL VORTICITY COMPRESSION AND OBSTRUCTION TO STEADY FLOW IN IDEAL FLUIDS

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1. Appendix I

We consider the invariance of helicity

$$H = \int_\Omega \theta \wedge d\theta, \tag{1}$$

under an arbitrary transformation, $\theta \to \theta + \beta$, where β is a closed 1-form. Then the helicity integral gives a boundary term

$$H \to \int_\Omega \theta \wedge d\theta + \int_{\partial\Omega} \theta \wedge \beta, \tag{2}$$

which in general does not vanish. By construction, $d\theta|_{\partial\Omega} = 0$, hence $\theta|_{\partial\Omega}$ is a closed 1-form on $\partial\Omega$ and defines a de Rahm cohomology class $[\theta] \in H^1(\partial\Omega; \mathbb{R})$. Restriction to the boundary defines a map $r : H^1(\Omega; \mathbb{R}) \to H^1(\partial\Omega; \mathbb{R})$. $d\theta$ is fluxless if $[\theta] = 0 \in H^1(\partial\Omega; \mathbb{R})/\mathrm{Im}(r)$. This condition is equivalent to the statement that

$$\int_S d\theta = 0, \tag{3}$$

for any surface $S \subset \Omega$, with $\partial S \subset \partial\Omega$.

2. Appendix II

We start with the definition

$$H = \frac{1}{U \cdot A} \omega \times U = \beta\omega \times U. \tag{4}$$

Then we assert that

$$(\partial_t + \mathcal{L}_U) H = fA, \tag{5}$$

where f is to be determined. Using coordinate notation (recall we are in Euclidean space, so that we do not distinguish covariant and contravariant indices), we have

$$fA_i = \partial_t H_i + U_j \partial_j H_i + U_j \partial_i H_j. \tag{6}$$

Now by construction we have $U_i H_i = 0$, so this is rewritten as

$$fA_i = \partial_t H_i + U_j \partial_j H_i - H_j \partial_i U_j, \tag{7}$$

or

$$fA = \partial_t H - U \times \nabla \times H. \tag{8}$$

Expanding we find

(9)
$$fA = (\partial_t\beta)\omega\times U+\beta(\partial_t\omega)\times U+\beta\omega\times(\partial_t U)-U\times\big(\nabla\beta\times(\omega\times U)\big)+\beta U\times(\partial_t\omega),$$

(recall $\beta = (U\cdot A)^{-1}$) which becomes

$$fA = ((\partial_t + U\cdot\nabla)\beta)\omega\times U + \beta\omega\times(\partial_t U). \tag{10}$$

Now, using the fact that

$$(\partial_t + U\cdot\nabla)A + (\nabla A)\cdot U = 0, \tag{11}$$

we find

$$((\partial_t + U\cdot\nabla)\beta) = \beta(H\cdot A + \beta A\cdot\nabla(P + U^2/2)). \tag{12}$$

So we get

(13)
$$fA = \big((h\cdot A) + \beta A\cdot\nabla(P+U^2/2)\big)\,H - \beta\omega\times\nabla(P+U^2/2) - \beta\omega\times(\omega\times U).$$

Then we find

$$fA = \big(H^2 + \beta H\cdot\nabla(P+U^2/2)\big)\,A, \tag{14}$$

so we may identify

$$f = H^2 + \beta H\cdot\nabla(P+U^2/2). \tag{15}$$