Characterizing reservoirs of infection and the maintenance of pathogens in ecosystems: Supplementary information

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1 The next generation matrix

The next generation matrix (NGM) $\mathbf{K}$ has entries K_{ij}, the expected number of hosts of type i that would be infected with pathogen P by a single infected host of type j over its entire infectious period, and in a fully susceptible and otherwise uninfected ecosystem [1, 2]. By definition $\mathbf{K}$ is a non-negative matrix, $K_{ij} \geqslant 0$ for all i and j. The basic reproduction number is the spectral radius (largest eigenvalue) of the NGM, $\mathcal{R}_0 = \rho(\mathbf{K})$. If there is an infection pathway from every host species to every other host species, then the NGM is irreducible. A matrix $\mathbf{K}$ is reducible if there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{K}\mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{B}$ and $\mathbf{D}$ are square matrices of order at least one. In the static analysis, removing or ignoring a host type replaces a row and column of $\mathbf{K}$ with zeroes, as a consequence $\mathcal{R}_{\Omega_m} \geqslant \mathcal{R}_{\Omega_{m-1}}$. This follows from Corollary 1.5 on page 27 of [3], and also by applying Gelfand's formula: for any matrix norm

$$\rho(\mathbf{K}) = \lim_{k\to\infty} \left\| \mathbf{K}^k \right\|^{1/k}$$

and choosing, for example, $\|\mathbf{K}\| = \max_{i,j} |K_{ij}|$.

2 Parameter values used in the examples

2.1 Parameter values used in Example 1: A prey predator system

The population dynamics are governed by

$$\boldsymbol{\phi} = \begin{pmatrix} 0.1500 & 0.1125 & 0.2500 & 0.2500 \\ 0.0750 & 0.1500 & 0.4000 & 0.1000 \\ -0.0750 & -0.1600 & 0.5000 & 0.0750 \\ -0.0750 & -0.0100 & 0.0750 & 0.3000 \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} 1.000 \\ 0.800 \\ 0.100 \\ 0.075 \end{pmatrix}$$

and $\boldsymbol{\nu}$ chosen so that $\mathbf{n}^* = \mathbf{1}$, leading to $\nu = (1.7625, 1.5250, 0.4400, 0.3650)'$.

The epidemiological parameters are

$$\boldsymbol{\gamma} = \text{diag} \begin{pmatrix} 0.75 \\ 0.50 \\ 0.15 \\ 0.10 \end{pmatrix} \qquad \boldsymbol{\kappa} = \begin{pmatrix} 0.35 \\ 0.75 \\ 0.50 \\ 1.00 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 0.5 \\ 0.4 \\ 0.1 \\ 0.2 \end{pmatrix}$$

with $\beta_{ij} = \kappa_i r_j$. Transmission via feeding: $c_{ji} = 2$ for $i, j = 1, 2$.

2.2 Parameter values used in Example 2: Five species types in competition

The population dynamics are governed by

$$\boldsymbol{\phi} = \begin{pmatrix} 0.4923 & 0.4192 & 0.6705 & 0.8007 & 0.8946 \\ 0.1863 & 1.0852 & 0.4173 & 0.9683 & 0.0850 \\ 0.3456 & 0.2045 & 0.9587 & 0.3134 & 0.0391 \\ 0.3968 & 0.8781 & 0.1404 & 1.0923 & 0.1698 \\ 0.5388 & 0.0274 & 0.1981 & 0.8764 & 1.2781 \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} 0.4170 \\ 0.7203 \\ 0.0001 \\ 0.3023 \\ 0.1468 \end{pmatrix}$$

and $\boldsymbol{\nu}$ chosen so that $\mathbf{n}^* = \mathbf{1}$, leading to

$$\nu = (3.6944, 3.4624, 1.8613, 2.9798, 3.0656)'$$

The epidemiological parameters are

$$\boldsymbol{\gamma} = \text{diag}\begin{pmatrix} 0.1967 \\ 0.8422 \\ 1.9158 \\ 1.0663 \\ 1.3838 \end{pmatrix} \qquad \boldsymbol{\kappa} = \begin{pmatrix} 0.6310 \\ 1.3730 \\ 1.6693 \\ 0.0366 \\ 1.5003 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 0.9889 \\ 0.7482 \\ 0.2804 \\ 0.7893 \\ 0.1032 \end{pmatrix}$$

with $\beta_{ij} = \kappa_i r_j$. Increased mortality due to infection is

$$\alpha = (0.75, 0.75, 0.75, 0.75)'$$

2.3 Parameter values used in Example 3: Five host species and five non-host species in competition

The population dynamics are governed by

$$\boldsymbol{\phi} = \begin{pmatrix} 1.3194 & 0.4361 & 0.5279 & 0.3969 & 0.8955 & 0.9047 & 0.7326 & 0.0485 & 0.6669 & 0.2609 \\ 0.1977 & 1.4890 & 0.9376 & 0.6239 & 0.3629 & 0.4048 & 0.8947 & 0.1098 & 0.1487 & 0.4125 \\ 0.8630 & 0.7863 & 1.0617 & 0.7674 & 0.6376 & 0.3312 & 0.5147 & 0.6840 & 0.3646 & 0.4190 \\ 0.9834 & 0.8663 & 0.1082 & 0.7190 & 0.1914 & 0.5721 & 0.6036 & 0.5147 & 0.8658 & 0.9024 \\ 0.1638 & 0.1732 & 0.1582 & 0.3756 & 1.0378 & 0.8454 & 0.0651 & 0.5716 & 0.3503 & 0.9796 \\ 0.5973 & 0.0749 & 0.5452 & 0.5025 & 0.1824 & 1.4010 & 0.5401 & 0.8437 & 0.1890 & 0.6236 \\ 0.0090 & 0.6007 & 0.5244 & 0.6867 & 0.9184 & 0.5957 & 0.6692 & 0.4877 & 0.4726 & 0.0832 \\ 0.3866 & 0.1680 & 0.6376 & 0.2537 & 0.4318 & 0.0847 & 0.6146 & 1.3501 & 0.3928 & 0.7330 \\ 0.0442 & 0.7334 & 0.4015 & 0.5547 & 0.8302 & 0.5973 & 0.3637 & 0.5102 & 1.1589 & 0.6787 \\ 0.9567 & 0.4084 & 0.6498 & 0.6249 & 0.4168 & 0.2455 & 0.7678 & 0.9267 & 0.4368 & 1.3660 \end{pmatrix}$$

$\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ chosen so that $\mathbf{n}^* = \mathbf{1}$, leading to

$$\boldsymbol{\mu} = \begin{pmatrix} 0.9670 \\ 0.5472 \\ 0.9727 \\ 0.7148 \\ 0.6977 \\ 0.2161 \\ 0.9763 \\ 0.0062 \\ 0.2530 \\ 0.4348 \end{pmatrix} \qquad \boldsymbol{\nu} = \begin{pmatrix} 7.1565 \\ 6.1287 \\ 7.4022 \\ 7.0417 \\ 5.4184 \\ 5.7159 \\ 6.0239 \\ 5.0590 \\ 6.1257 \\ 7.2341 \end{pmatrix}$$

The epidemiological parameters are

$$\boldsymbol{\gamma} = \mathrm{diag} \begin{pmatrix} 0.6080 \\ 0.1030 \\ 1.0707 \\ 0.2170 \\ 1.3292 \end{pmatrix} \qquad \boldsymbol{\kappa} = \begin{pmatrix} 1.3902 \\ 0.7151 \\ 1.6516 \\ 0.3041 \\ 1.6495 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 0.8146 \\ 1.3352 \\ 1.3174 \\ 1.7171 \\ 1.7032 \end{pmatrix}$$

with $\beta_{ij} = \kappa_i r_j$ for $i, j = 1 \ldots 5$ and $\beta_{ij} = 0$ for $i, j = 6 \ldots 10$.

3 Results from the examples

3.1 Results from Example 1: A prey predator system

m	Ω_m	$\mathcal{R}_{\Omega_m}$	$\mathcal{R}_0(\Omega_m)$
4	$\{1,2,3,4\}$	1.4007	1.4007
3	$\{2,3,4\}$	1.2577	1.2798
3	$\{1,3,4\}$	1.1295	1.3350
3	$\{1,2,4\}$	1.2264	1.8439
3	$\{1,2,3\}$	0.8947	0.9586
2	$\{3,4\}$	0.9481	0.7999
2	$\{2,4\}$	1.0797	1.8208
2	$\{2,3\}$	0.7763	0.9702
2	$\{1,4\}$	1.0010	1.5250
2	$\{1,3\}$	0.6488	0.9308
2	$\{1,2\}$	0.6645	1.3065
1	$\{4\}$	0.8219	0.8037
1	$\{3\}$	0.4545	0.4182
1	$\{2\}$	0.5246	1.2787
1	$\{1\}$	0.5248	0.9303

Table S1: These results are illustrated in Figure 2

3.2 Results from Example 2: Five species types in competition

m	Ω_m	$\mathcal{R}_{\Omega_m}$	$\mathcal{R}_0(\Omega_m)$
5	$\{1,2,3,4,5\}$	1.0783	1.0783
4	$\{2,3,4,5\}$	1.0184	0.9827
4	$\{1,2,3,5\}$	1.0736	1.1875
3	$\{3,4,5\}$	0.9293	0.9630
3	$\{2,4,5\}$	0.5335	0.6821
3	$\{2,3,5\}$	1.0145	1.2656
3	$\{2,3,4\}$	0.9962	0.9579
3	$\{1,2,5\}$	0.6127	1.2142
2	$\{4,5\}$	0.4060	0.3955
2	$\{3,5\}$	0.9268	1.1078
2	$\{3,4\}$	0.9152	0.9528
2	$\{2,5\}$	0.5251	0.8919
2	$\{2,4\}$	0.4555	0.5089
2	$\{2,3\}$	0.9926	1.2371
2	$\{1,4\}$	0.3028	0.9612
2	$\{1,2\}$	0.5401	1.2396
1	$\{5\}$	0.4032	0.4553
1	$\{4\}$	0.2936	0.3049
1	$\{3\}$	0.9129	1.0817
1	$\{2\}$	0.4438	0.8161
1	$\{1\}$	0.1847	0.9789

Table S2: These results are illustrated in Figure 3

3.3 Results from Example 3: Five host species and five non-host species in competition

m	Ω_m	$\mathcal{R}_{\Omega_m}$	$\mathcal{R}_0(\Omega_m)$	
$8g$	$\{1,3,4,6,7,8,9,10\}$	0.6401	0.8899	green
$8y$	$\{1,2,5,6,7,8,9,10\}$	1.0149	0.9182	yellow
$8r1$	$\{1,2,3,5,6,8,9,10\}$	1.2974	1.1676	red
$8r2$	$\{1,2,3,4,5,6,9,10\}$	1.3635	1.3357	red
$7y$	$\{2,3,4,5,6,9,10\}$	1.2171	0.8954	yellow
$7b1$	$\{1,3,4,6,8,9,10\}$	0.6401	1.0493	blue
$7r$	$\{1,2,5,6,7,8,10\}$	1.0149	1.1718	red
$7g$	$\{1,2,3,6,8,9,10\}$	0.7087	0.8156	green
$7b2$	$\{1,2,3,4,6,9,10\}$	0.7755	1.1901	blue
$6r1$	$\{2,3,5,6,9,10\}$	1.1529	1.5726	red
$6y$	$\{2,3,4,5,6,10\}$	1.2171	0.8176	yellow
$6g$	$\{1,3,4,6,8,9\}$	0.6401	0.9486	green
$6r2$	$\{1,2,5,6,8,10\}$	1.0149	1.3529	red
$6b$	$\{1,2,3,6,9,10\}$	0.7087	1.2483	blue
$5g1$	$\{2,3,4,6,10\}$	0.6213	0.7804	green
$5b$	$\{1,3,4,8,9\}$	0.6401	1.0387	blue
$5g2$	$\{1,2,6,9,10\}$	0.3688	0.6453	green
$5y$	$\{1,2,5,6,10\}$	1.0149	0.8087	yellow

Table S3: These results are illustrated in Figure 4

References

[1] Diekmann, O., Heesterbeek, J. A. P., Britton, T., 2013. Mathematical tools for understanding infectious disease dynamics. Princeton University Press, Princeton.

[2] Diekmann, O., Heesterbeek, J. A. P., Roberts, M. G., 2010. The construction of next-generation matrices for compartmental epidemic models. Journal of the Royal Society Interface 7, 873-885.

[3] Berman, A., Plemmons, R. J., 1979. Nonnegative matrices in the mathematical sciences. Academic Press, New York.