Supplementary 1: Expression for $\mathtt{u}_{\ \ 1,\mathrm{N}}$

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Equation (3.34) implies $\mathtt{u}_{\mathrm{N};\ \ } = \mathcal{Q}^{-1}(-\mathcal{W}_{\mathrm{N}} + \mathtt{u}_{\mathrm{N}+1;-} + \mathtt{u}_{\mathrm{N}-1;-}) = \mathcal{Q}^{-1}(-\mathcal{W}_{\mathrm{N}} + \mathtt{w}_{\mathrm{N};-})$. Using the inverse discrete Fourier transform (3.3),

$$\mathtt{u}_{-1,\mathrm{N}} = \frac{1}{2\pi i}\oint_{\mathscr{C}} \mathtt{u}_{\mathrm{N};-}(z)z^{-1-1}dz = \frac{1}{2\pi i}\oint_{\mathscr{C}} \frac{-\mathcal{W}_{\mathrm{N}}(z) + \mathtt{w}_{\mathrm{N};-}(z)}{\mathcal{Q}(z)} z^{-2}dz. \tag{A 4}$$

With $\mathcal{Q}(z) = z_q^{-1}(1 - z_q z)(1 - z_q z^{-1}) = -z^{-1}(z - z_q)(z - z_q^{-1})$, above implies

$$\begin{aligned}\mathtt{u}_{-1,\mathrm{N}} &= \frac{1}{2\pi i}\oint_{\mathscr{C}} \frac{-\mathcal{W}_{\mathrm{N}}(z) + \mathtt{w}_{\mathrm{N};-}(z)}{-(z - z_q)(z - z_q^{-1})} z^{-1}dz = (\mathcal{W}_{\mathrm{N}}(z_q) - \mathtt{w}_{\mathrm{N};-}(z_q))z_q^{-1} + (\mathcal{W}_{\mathrm{N}}(0) - \mathtt{w}_{\mathrm{N};-}(0))\\ &= (\mathtt{u}_{-1,\mathrm{N}} + z_q\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}} - \mathtt{w}_{\mathrm{N};-}(z_q))z_q^{-1} + (\mathtt{u}_{-1,\mathrm{N}} - 0),\end{aligned} \tag{A 5}$$

where $\mathtt{w}_{\mathrm{N};-}(z_q) = \mathcal{C}_-(z_q)\mathcal{L}_-(z_q)$

$$\begin{aligned}&= \mathcal{L}_-(z_q)(\mathtt{u}_{-1,\mathrm{N}}(\mathcal{L}_-^{-1}(z_q) - \bar{l}_{-0}) + z_q\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}(\mathcal{L}_-^{-1}(z_q) - l_{+0}) + \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}\delta_{D+}(z_q z_{\mathrm{P}}^{-1})\\ &\big(\mathcal{Q}(z_q)\mathcal{L}_-^{-1}(z_q) - \mathcal{Q}(z_{\mathrm{P}})\mathcal{L}_-^{-1}(z_{\mathrm{P}}) + \bar{l}_{-0}(z_q^{-1} - z_{\mathrm{P}}^{-1}) + l_{+0}(z_q - z_{\mathrm{P}})\big))\\ &= \mathtt{u}_{-1,\mathrm{N}}(1 - \mathcal{L}_-(z_q)\bar{l}_{-0}) + z_q\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}(1 - \mathcal{L}_-(z_q)l_{+0}) + \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}\delta_{D+}(z_q z_{\mathrm{P}}^{-1})\\ &\big(\mathcal{Q}(z_q) - \mathcal{L}_-(z_q)\mathcal{Q}(z_{\mathrm{P}})\mathcal{L}_-^{-1}(z_{\mathrm{P}}) + \bar{l}_{-0}(z_q^{-1} - z_{\mathrm{P}}^{-1})\mathcal{L}_-(z_q) + l_{+0}(z_q - z_{\mathrm{P}})\mathcal{L}_-(z_q)\big).\end{aligned}$$

Hence,

$$\begin{aligned}0 &= \mathtt{u}_{-1,\mathrm{N}}(-\mathcal{L}_-(z_q)\bar{l}_{-0}) + z_q\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}(-\mathcal{L}_-(z_q)l_{+0}) + \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}\frac{z_q}{z_q - z_{\mathrm{P}}}\\ &\big(\mathcal{Q}(z_q) - \mathcal{L}_-(z_q)\mathcal{Q}(z_{\mathrm{P}})\mathcal{L}_-^{-1}(z_{\mathrm{P}}) + \bar{l}_{-0}(z_q^{-1} - z_{\mathrm{P}}^{-1})\mathcal{L}_-(z_q) + l_{+0}(z_q - z_{\mathrm{P}})\mathcal{L}_-(z_q)\big)\\ &= -\mathtt{u}_{-1,\mathrm{N}}\mathcal{L}_-(z_q)\bar{l}_{-0} - \frac{\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}z_q}{z_q - z_{\mathrm{P}}}\mathcal{L}_-(z_q)\mathcal{Q}(z_{\mathrm{P}})\mathcal{L}_-^{-1}(z_{\mathrm{P}}) - \bar{l}_{-0}z_{\mathrm{P}}^{-1}\mathcal{L}_-(z_q)\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}},\end{aligned}$$

which gives $\mathtt{u}_{-1,\mathrm{N}} = \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}}(-\frac{z_q}{z_q - z_{\mathrm{P}}}\frac{\mathcal{Q}(z_{\mathrm{P}})}{\bar{l}_{-0}\mathcal{L}_-(z_{\mathrm{P}})} - z_{\mathrm{P}}^{-1})$, i.e., (3.46) holds. Similarly, in the case of *incidence from the waveguide*, (3.50) implies

$$\mathcal{C}_-(z) = \mathtt{u}_{-1,\mathrm{N}}(\mathcal{L}_-^{-1}(z) - \bar{l}_{-0}) - \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}\,-1}\delta_{D-}(z z_{\mathrm{P}}^{-1})\big(\mathcal{L}_-^{-1}(z) - \mathcal{L}_+(z_{\mathrm{P}})\big), \tag{A 6}$$

so that by (3.49),

$$\begin{aligned}\mathtt{u}_{-1,\mathrm{N}} &= \frac{1}{2\pi i}\oint_{\mathscr{C}} \frac{-\mathcal{W}_{\mathrm{N}}(z) + \mathtt{w}_{\mathrm{N};-}(z) + \mathtt{w}^{\mathrm{i}}_{\mathrm{N};-}(z)}{-z^{-1}(z - z_q)(z - z_q^{-1})} z^{-2}dz\\ &= (\mathcal{W}_{\mathrm{N}}(z_q) - \mathtt{w}_{\mathrm{N};-}(z_q) - \mathtt{w}^{\mathrm{i}}_{\mathrm{N};-}(z_q))z_q^{-1}/(z_q - z_q^{-1}) + (\mathcal{W}_{\mathrm{N}}(0) - \mathtt{w}_{\mathrm{N};-}(0) - \mathtt{w}^{\mathrm{i}}_{\mathrm{N};-}(0))\\ &= (\mathtt{u}_{-1,\mathrm{N}} + z_q\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}} - \mathtt{w}_{\mathrm{N};-}(z_q) - \mathtt{w}^{\mathrm{i}}_{\mathrm{N};-}(z_q))z_q^{-1}/(z_q - z_q^{-1}) + (\mathtt{u}_{-1,\mathrm{N}} - 0),\end{aligned} \tag{A 7}$$

where $\mathtt{w}_{\mathrm{N};-}(z_q) = \mathcal{C}_-(z_q)\mathcal{L}_-(z_q)$

$$\begin{aligned}&= \mathcal{L}_-(z_q)(\mathtt{u}_{-1,\mathrm{N}}(\mathcal{L}_-^{-1}(z_q) - \bar{l}_{-0}) - \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}\,-1}\delta_{D-}(z_q z_{\mathrm{P}}^{-1})\big(\mathcal{L}_-^{-1}(z_q) - \mathcal{L}_+(z_{\mathrm{P}})\big))\\ &= \mathtt{u}_{-1,\mathrm{N}}(1 - \mathcal{L}_-(z_q)\bar{l}_{-0}) - \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}\,-1}\delta_{D-}(z_q z_{\mathrm{P}}^{-1})\big(1 - \mathcal{L}_-(z_q)\mathcal{L}_+(z_{\mathrm{P}})\big)).\end{aligned}$$

Hence, $0 = -\mathtt{u}_{-1,\mathrm{N}}(-\mathcal{L}_-(z_q)\bar{l}_{-0}) - \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}\,-1}\frac{z_q}{z_q - z_{\mathrm{P}}} + \mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}\,-1}\frac{z_q}{z_q - z_{\mathrm{P}}}(1 - \mathcal{L}_-(z_q)\mathcal{L}_+(z_{\mathrm{P}}))$, which gives $\mathtt{u}_{-1,\mathrm{N}} = -\mathtt{u}^{\mathrm{i}}_{0,\mathrm{N}\,-1}\frac{z_q}{z_q - z_{\mathrm{P}}}\frac{\mathcal{L}_+(z_{\mathrm{P}})}{\bar{l}_{-0}}$, i.e., (3.51) holds.