```
% Figure 4
% Plotting output from viscfoot3.m
%
load('visc_foot3_n_1_3_ddot2.mat') % from viscfoot3.m


% Free surface and yield surface
% indenter plot
figure(1)
set(gcf,'position',[100 100 600 300])
subplot('position',[0.06 0.60 0.3 0.39])
hold on
for i = 21;%:3:n1;

    z = y(i,:);


    nd = length(h(i,:));
    for j = 1:nd-1;
        dh(j) = (h(i,j+1)-h(i,j))/dx;
    end
    dh(nd) = 0;

    Y = (h(i,:) - B./(R*(sqrt(((dh)).^2+epsilon^2))));
    [~,X,Px] = visc_foot3_pressure(t,z,V,B,R,n,epsilon,Epsilon,M,dx);
    Y1 = ((y(i,m-1)+(1/2)*X.^2)/2)+(B./(Px+R*X));
    Y2 = ((y(i,m-1)+(1/2)*X.^2)/2)-(B./(Px+R*X));

    z = h(i,:);
    eta(i) = visc_foot3_eta(L(i));
    plot((0:0.01:4),delta(i)+(1/2)*(0:0.01:4).^2,'k')
    area((0:0.01:4),[2 2*(0.01:0.01:4)./(0.01:0.01:4)],'FaceColor',[222 222
227]/255)
    area((0:0.01:4),delta(i)+(1/2)*(0:0.01:4).^2,'FaceColor','w')


    area([x+L(i) (x(end)+L(i):0.1:2)],[h(i,:)
(x(end)+L(i):0.1:2)./(x(end)+L(i):0.1:2)],'FaceColor','b','EdgeColor','none
','FaceAlpha',0.5)
    area(x+L(i),Y,'FaceColor','w')
    plot(0,delta(i),'k^','MarkerFaceColor','k')
    for j = 1:length(h(i,:));
        if h(i,j) < 1.01;
            Ln = L(i)+j*dx;
            break
        end
    end
    plot([x+L(i) (1.5:0.1:1.8)],[h(i,:) (1.5:0.1:1.8)./(1.5:0.1:1.8)],'b')
    plot(Ln*[1 (0.1:0.1:1)./(0.1:0.1:1)],(0:0.1:1),'k--')
    plot(Ln,0,'ks','MarkerFaceColor','b')

    area(X,Y2,'FaceColor',[255 150 139]/255,'FaceAlpha',0.5)
    area(X,Y1,'FaceColor','w')
    plot(X,Y1,'r')
    plot(X,Y2,'r')
    plot(x+L(i),Y,'b')


end
plot((0:0.1:1.8),1.6*[1 (0.1:0.1:1.8)./(0.1:0.1:1.8)],'k')
plot(X(end)*(0:0.01:h(i,1))./(0:0.01:h(i,1)),(0:0.01:h(i,1)),'w')
```

```
plot(X(end)*(0:0.01:h(i,1))./(0:0.01:h(i,1)),(0:0.01:h(i,1)),'k--')
plot(X(end),0,'ko','MarkerFaceColor','r')

% xlabel('$x$','Interpreter','latex','FontSize',12)
ylabel('$z$','Interpreter','latex','FontSize',14)
text(0.07,1.45,'(a) $t=0.02$','Interpreter','latex','FontSize',14)
axis([0 1.8 0 1.6])
box on

subplot('position',[0.37 0.60 0.3 0.39])
hold on
for i = 151;%:3:n1;

    z = y(i,:);

    nd = length(h(i,:));
    for j = 1:nd-1;
        dh(j) = (h(i,j+1)-h(i,j))/dx;
    end
    dh(nd) = 0;

    Y = (h(i,:) - B./(R*(sqrt(((dh)).^2+epsilon^2))));
    [~,X,Px] = visc_foot3_pressure(t,z,V,B,R,n,epsilon,Epsilon,M,dx);
    Y1 = ((y(i,m-1)+(1/2)*X.^2)/2)+(B./(Px+R*X));
    Y2 = ((y(i,m-1)+(1/2)*X.^2)/2)-(B./(Px+R*X));

    z = h(i,:);
    eta(i) = visc_foot3_eta(L(i));
    plot((0:0.01:4),delta(i)+(1/2)*(0:0.01:4).^2,'k')
    area((0:0.01:4),[2 2*(0.01:0.01:4)./(0.01:0.01:4)],'FaceColor',[222 222
227]/255)
    area((0:0.01:4),delta(i)+(1/2)*(0:0.01:4).^2,'FaceColor','w')


    area([x+L(i) (x(end)+L(i):0.1:2)],[h(i,:)
(x(end)+L(i):0.1:2)./(x(end)+L(i):0.1:2)],'FaceColor','b','EdgeColor','none
','FaceAlpha',0.5)
    area(x+L(i),Y,'FaceColor','w')
    plot(0,delta(i),'k^','MarkerFaceColor','k')
    for j = 1:length(h(i,:));
        if h(i,j) < 1.01;
            Ln = L(i)+j*dx;
            break
        end
    end
    plot([x+L(i) (1.5:0.1:1.8)],[h(i,:) (1.5:0.1:1.8)./(1.5:0.1:1.8)],'b')
    plot(Ln*[1 (0.1:0.1:1)./(0.1:0.1:1)],(0:0.1:1),'k--')
    plot(Ln,0,'ks','MarkerFaceColor','b')

    area(X,Y2,'FaceColor',[255 150 139]/255,'FaceAlpha',0.5)
    area(X,Y1,'FaceColor','w')
    plot(X,Y1,'r')
    plot(X,Y2,'r')
    plot(x+L(i),Y,'b')


end
plot((0:0.1:1.8),1.6*[1 (0.1:0.1:1.8)./(0.1:0.1:1.8)],'k')
plot(X(end)*(0:0.01:h(i,1))./(0:0.01:h(i,1)),(0:0.01:h(i,1)),'w')
plot(X(end)*(0:0.01:h(i,1))./(0:0.01:h(i,1)),(0:0.01:h(i,1)),'k--')
plot(X(end),0,'ko','MarkerFaceColor','r')
```

```
xlabel('$x$','Interpreter','latex','FontSize',14)
% ylabel('$y$','Interpreter','latex','FontSize',12)
set(gca,'ytick',[],'yticklabels',[])
text(0.07,1.45,'(b) $t=0.15$','Interpreter','latex','FontSize',14)
axis([0 1.8 0 1.6])
box on

subplot('position',[0.68 0.60 0.3 0.39])
hold on
for i = n1;%:3:n1;

    z = y(i,:);

    nd = length(h(i,:));
    for j = 1:nd-1;
        dh(j) = (h(i,j+1)-h(i,j))/dx;
    end
    dh(nd) = 0;

    Y = (h(i,:) - B./(R*(sqrt(((dh)).^2+epsilon^2))));
    [~,X,Px] = visc_foot3_pressure(t,z,V,B,R,n,epsilon,Epsilon,M,dx);
    Y1 = ((y(i,m-1)+(1/2)*X.^2)/2)+(B./(Px+R*X));
    Y2 = ((y(i,m-1)+(1/2)*X.^2)/2)-(B./(Px+R*X));

    z = h(i,:);
    eta(i) = visc_foot3_eta(L(i));
    plot((0:0.01:4),delta(i)+(1/2)*(0:0.01:4).^2,'k')
    area((0:0.01:4),[2 2*(0.01:0.01:4)./(0.01:0.01:4)],'FaceColor',[222 222
227]/255)
    area((0:0.01:4),delta(i)+(1/2)*(0:0.01:4).^2,'FaceColor','w')


    area([x+L(i) (x(end)+L(i):0.1:2)],[h(i,:)
(x(end)+L(i):0.1:2)./(x(end)+L(i):0.1:2)],'FaceColor','b','EdgeColor','none
','FaceAlpha',0.4)
    area(x+L(i),Y,'FaceColor','w')
    plot(0,delta(i),'k^','MarkerFaceColor','k')
    for j = 1:length(h(i,:));
        if h(i,j) < 1.01;
            Ln(i) = L(i)+j*dx;
            break
        end
    end
    plot([x+L(i)],[h(i,:)],'b')
    plot(Ln(i)*[1 (0.1:0.1:1)./(0.1:0.1:1)],(0:0.1:1),'k--')
    plot(Ln(i),0,'ks','MarkerFaceColor','b')

    area(X,Y2,'FaceColor',[255 150 139]/255,'FaceAlpha',0.5)
    area(X,Y1,'FaceColor','w')
    plot(X,Y1,'r')
    plot(X,Y2,'r')
    plot(x+L(i),Y,'b')


end
plot((0:0.1:1.8),1.6*[1 (0.1:0.1:1.8)./(0.1:0.1:1.8)],'k')
plot(X(end)*(0:0.01:h(i,1))./(0:0.01:h(i,1)),(0:0.01:h(i,1)),'w')
plot(X(end)*(0:0.01:h(i,1))./(0:0.01:h(i,1)),(0:0.01:h(i,1)),'k--')
plot(X(end),0,'ko','MarkerFaceColor','r')
% xlabel('$x$','Interpreter','latex','FontSize',12)
% ylabel('$y$','Interpreter','latex','FontSize',12)
```

```
set(gca,'ytick',[],'yticklabels',[])
text(0.07,1.45,'(c) $t=0.296$','Interpreter','latex','FontSize',14)
axis([0 1.8 0 1.6])
box on

eq = load('visc_foot3_equil.mat'); % from visc_foot3_dequil.m

del = eq.del;
lam = eq.lam;
[~,ind] = min(abs(delta-del));


% Depth of the indenter
subplot('position',[0.065 0.105 0.42 0.36])
hold on
plot(tspan(1:n1),1-delta,'k')
plot(tspan([21 151]),1-delta([21
151]),'k^','MarkerFaceColor','k','MarkerSize',5)
plot(tspan(n1),1-delta(n1),'k^','MarkerFaceColor','k','MarkerSize',5)
plot(tspan(ind),1-del,'kp','MarkerSize',5,'MarkerFaceColor','k')
ylabel('$\delta (t)$','Interpreter','latex','FontSize',14)
xlabel('$t$','Interpreter','latex','FontSize',14)
text(0.27,0.05,'$(d)$','Interpreter','latex','FontSize',14)
text(0.05,0.18,'$\delta_{eq}$','Interpreter','latex','FontSize',14)
box on
axis([0 0.3 0 .4])

for i = 1:n1;
    for j = 1:length(h(i,:));
        if h(i,j) < 1.01;
            Ln(i) = L(i)+j*dx;
            break
        end
    end
end

subplot('position',[0.56 0.105 0.42 0.36])
hold on
plot(tspan(1:n1),L,'r')
h1 = plot(tspan([21 151 n1]),L([21 151
n1]),'ko','MarkerFaceColor','r','MarkerSize',5);
plot(tspan(1:n1),Ln,'b')
h2 = plot(tspan([21 151 n1]),Ln([21 151
n1]),'ks','MarkerFaceColor','b','MarkerSize',5);
ylabel('$\xi_C (t), \, \xi_E (t)$','Interpreter','latex','FontSize',14)
xlabel('$t$','Interpreter','latex','FontSize',14)
legend([h1 h2],{'$\xi_C (t)$','$\xi_E
(t)$'},'Interpreter','latex','location','southeast',...
    'fontsize',14,'box','off')
text(0.015,(0.35/0.4)*2,'$(e)$','Interpreter','latex','FontSize',14)
box on

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3.m
% Viscoplastic indenter loading stage
% Squeeze flow and free surface
% Herschel-Bulkley fluid with parameter n
close all;clear all
%% Parameters
V = 1; % inertial/viscosity mux0^3/mh0^2(gh0)^(1/2)
B = 1; % Bingham/yield stress tau_Yx0^2/mgh0
```

```
R = 1; % gravity/density rhox0h0/m
n = 1/3; % n is the power-law index of Herschel-Bulkley fluid (n=1
perfectly plastic, n=0.3 Carbopol)
epsilon = 0.01;
Epsilon = 0.01;

%% Variables
dx = 2e-3;
x = linspace(0,0.5,0.5/dx);
M = length(x(1,:));
F = 1; % non-dimensional force applied by the indenter

%% Initial conditions
y0(1) = 1e-6; % contact point L
y0(2:M) = 1; % h(2:M)
y0(M+1) = 1; % delta
y0(M+2) = -2; % ddelta


%% System of coupled ODEs
tspan = 0:0.001:0.5;
options = odeset('RelTol',1e-6,'AbsTol',1e-
6,'Stats','off','OutputFcn',@odeplot,'Events',@visc_foot3_eventfun);
[t,y] = ode15s(@(t,y)
visc_foot3_ode(t,y,V,B,R,n,epsilon,Epsilon,M,dx),tspan,y0,options);

[n1,m] = size(y);
h = y(:,1:m-2);
L = y(:,1);
delta = y(:,m-1);
ddelta = y(:,m);

for i = 1:n1;
    z = h(i,:);
    eta(i) = visc_foot3_eta(z(1));
    h(i,1) = delta(i) + eta(i);
end


%% Conservation of volume
figure(3)
for i = 1:n1;
    V1(i) = trapz(x,h(i,:)-1);

    xn = linspace(0,4,8000);
    [~,ind] = min(abs(xn-L(i)));
    if ind > 1;
        V2(i) = trapz(xn(1:ind),(1/2)*xn(1:ind).^2);
    else
        V2(i) = 0;
    end

    V3(i) = L(i)*delta(i);

end
plot(t,V1,'b',t,V2,'r',t,V3,'g',t,V1+V2+V3,'k',t,L,'k--')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3_eventfun.m
function [value,isterminal,direction] = visc_foot3_eventfun(t,y)
```

```
% event function to decide whether to stop integration

    value = [y(end)+0.000001,y(end-1)-0.01];
    isterminal = 1;
    direction = 0;

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3_ode.m
function dydt = visc_foot3_ode(t,y,V,B,R,n,epsilon,Epsilon,M,dx)
% coupled system of odes

    Y1 = visc_foot3_yield(t,y,B,R,epsilon,Epsilon,M,dx);
    [N,~,~] = visc_foot3_pressure(t,y,V,B,R,n,epsilon,Epsilon,M,dx);
    eta = visc_foot3_eta(y(1));
    deta = visc_foot3_deta(y(1));

    U = -(1/dx)*(((n*R/((n+1)*((2*n)+1)*V))*(abs((y(2)-y(M+1)-
eta)/dx))^(1/n)*Y1(1)^(1+(1/n))*(((2*n)+1)*(y(M+1)+eta)-(n*Y1(1)))) +
(y(1)+dx)*y(M+2))/(deta-((y(2)-y(M+1)-eta)/dx));

    dydt = zeros(M,1);

    dydt(1) = U;
    dydt(2) = -(n*R/((n+1)*((2*n)+1)*V*dx))*((abs((y(3)-
y(2))/dx))^(1/n)*Y1(2)^(1+(1/n))*((((2*n)+1)*(y(3)+y(2))/2)-(n*Y1(2))))...
            +(n*R/((n+1)*((2*n)+1)*V*dx))*((abs((y(2)-
(y(M+1)+eta))/dx))^(1/n)*Y1(1)^(1+(1/n))*((((2*n)+1)*(y(2)+y(M+1)+eta)/2)-
(n*Y1(1))))...
            +(1/dx)*U*(y(3)-y(2));

    dydt(3:M-1) = -(n*R/((n+1)*((2*n)+1)*V*dx))*((abs((y(4:M)-y(3:M-
1))/dx)).^(1/n).*Y1(3:M-1).^(1+(1/n)).*((((2*n)+1)*(y(4:M)+y(3:M-1))/2)-
(n*Y1(3:M-1))))...
        +(n*R/((n+1)*((2*n)+1)*V*dx))*((abs((y(3:M-1)-y(2:M-
2))/dx)).^(1/n).*Y1(2:M-2).^(1+(1/n)).*((((2*n)+1)*(y(3:M-1)+y(2:M-2))/2)-
(n*Y1(2:M-2))))...
        +(1/dx)*U*(y(4:M)-y(3:M-1));

    dydt(M) = (n*R/((n+1)*((2*n)+1)*V*dx))*((abs((y(M)-y(M-
1))/dx))^(1/n)*Y1(M-1)^(1+(1/n))*((((2*n)+1)*(y(M)+y(M-1))/2)-n*Y1(M-1)));

    dydt(M+1) = y(M+2);
    dydt(M+2) = - 1 + N;


end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3_yield.m
function Y1 = visc_foot3_yield(t,y,B,R,epsilon,Epsilon,M,dx)
% yield surfaces

    eta = visc_foot3_eta(y(1));

    Y = zeros(M,1);
    Y1 = zeros(M,1);
    Y(1) = ((y(2)+y(M+1)+eta)/2) - B/(R*(sqrt(((y(2)-
(y(M+1)+eta))/dx)^2+epsilon^2)));
```

```
    Y(2:M-1) = ((y(3:M)+y(2:M-1))/2) - B./(R*(sqrt(((y(3:M)-y(2:M-
1))/dx).^2+epsilon^2)));
    Y(M) = 0;

    Y1(1:M) = (Y(1:M)+sqrt(Y(1:M).^2 + Epsilon^2))/2;

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3_eta.m
function eta = visc_foot3_eta(u)
% geometry of the indenter

    eta = (1/2)*u^2;

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3_pressure.m
function [N,xl,Px] = visc_foot3_pressure(t,y,V,B,R,n,epsilon,Epsilon,M,dx)
% pressure field P = P(delta,ddelta) underneath the indenter

    xl = 0:y(1)/50:y(1);
    Ml = length(xl);

    % quintic equation for Gamma (for n=1/3)
    px(1) = 2*B/y(M+1);
    Px(1) = -px(1);
    for i = 2:Ml;
%         xeta = xl(i);
        eta = visc_foot3_eta(xl(i));
        p = [(1/16)*(y(M+1)+eta)^5 (-15/32)*B*(y(M+1)+eta)^4
(5/4)*B^2*(y(M+1)+eta)^3 (-(5/4)*B^3*(y(M+1)+eta)^2-5*xl(i)*V*abs(y(M+2)))
0 (1/2)*B^5];
        % pressure gradient
        Pxi = roots(p);
        Px1(i) = Pxi(1);
        Px2(i) = Pxi(2);
        Px3(i) = Pxi(3);
        Px4(i) = Pxi(4);
        Px5(i) = Pxi(5);
            if (isreal(Px1(i)) == 1) && (Px1(i) >= (2*B/(y(M+1)+eta)));
                px(i) = Px1(i);
            elseif (isreal(Px2(i)) == 1) && (Px2(i) >= (2*B/(y(M+1)+eta)));
                px(i) = Px2(i);
            elseif (isreal(Px3(i)) == 1) && (Px2(i) >= (2*B/(y(M+1)+eta)));
                px(i) = Px3(i);
            elseif (isreal(Px4(i)) == 1) && (Px2(i) >= (2*B/(y(M+1)+eta)));
                px(i) = Px4(i);
            else
                px(i) = Px5(i);
            end
        deta = visc_foot3_deta(xl(i));
        Px(i) = -px(i) - R*deta; % subtract off gravity part

    end

    % integrate to get normal force, int_0^L P = -int_0^L x*Px
    N = -2*trapz(xl,Px.*xl);

end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot_deta.m
function deta = visc_foot3_deta(u)
% differentiate geometry
    deta = u;

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
visc_foot3_dequil.m
% Program to calculate and plot equilibrium position for visc_foot3_plot

clear
close all
G = 1;
B = 1;
d1 = 0.5:0.00001:0.95;

for i = 1:length(d1);
    p = [G/(2*B) 1 2*d1(i)*G/B -6*(1-d1(i)) (G/B)*(1+2*d1(i))*(d1(i)-1)];
    r = roots(p);

    if (isreal(r(1))==1) && (r(1)>0) && (r(1)<sqrt(6*(1-d1(i)))) && 
(r(1)>sqrt(2*(1-d1(i))))
        l(i) = r(1);
    elseif (isreal(r(2))==1) && (r(2)>0) && (r(2)<sqrt(6*(1-d1(i)))) && 
(r(2)>sqrt(2*(1-d1(i))))
            l(i) = r(2);
    elseif (isreal(r(3))==1) && (r(3)>0) && (r(3)<sqrt(6*(1-d1(i)))) && 
(r(3)>sqrt(2*(1-d1(i))))
            l(i) = r(3);
    elseif (isreal(r(4))==1) && (r(4)>0) && (r(4)<sqrt(6*(1-d1(i)))) && 
(r(4)>sqrt(2*(1-d1(i))))
            l(i) = r(4);
    end
end
l(end) = 0;

d2 = (l.^2/2)./(exp((1-(2/3)*G*l.^3)./(4*B))-1);

d = d1./d2;

[~,ind] = min(abs(d-1));
del = d1(ind); lam = l(ind);

figure(1)
hold on
plot(l,d1,'m')
plot(l,d2,'g')
figure(2)
hold on
plot(l,d,'b')
```