Supporting Information for "Analyzing Diffusion and Flow-driven Instability using Semidefinite Programming"

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Proof of Lemma 1

($\Rightarrow$) Let $\lambda_{\zeta,i}$ $(i = 1, 2, \cdots, n)$ denote the roots of the characteristic polynomial $\varphi(\zeta, s) = 0$, and suppose $\mathrm{Re}[\lambda_{\zeta,i}] < 0$ for all $\zeta \in \mathcal{Z}$, Solving equation (6), we have

$$\tilde{\boldsymbol{c}}(\zeta, t) = \sum_{i=1}^{n} h_i e^{\lambda_{\zeta,i} t} \boldsymbol{v}_i(\zeta, 0), \tag{A.1}$$

where $\boldsymbol{v}_i(\zeta, 0)$ $(i = 1, 2, \cdots, n)$ are eigenvectors of $A - \zeta^2 D + j\zeta V$, and h_i $(i = 1, 2, \ldots, n)$ are constants determined from a given initial condition $\tilde{\boldsymbol{c}}(\zeta, 0)$. Since the roots of $\lim_{\zeta\to\infty} \varphi(\zeta, s) = 0$ converge to those of $|sI + D| = 0$, which are $-d_i < 0$ $(i = 1, 2, \cdots, n)$ due to the continuity of the polynomial roots, the asymptotes do not converge to the imaginary axis. Thus, there exists $\gamma > 0$ such that $\max_\zeta \max_i \mathrm{Re}[\lambda_{\zeta,i}] < -\gamma$. Moreover, it follows that

$$\begin{aligned}\int_\Omega \|\boldsymbol{c}(x,t)\| dx &= \int_\Omega \left\| \frac{1}{2\pi} \sum_{\zeta\in\mathcal{Z}} \tilde{\boldsymbol{c}}(\zeta, t) e^{j\zeta x} \right\| dx \\ &= \int_\Omega \left\| \sum_{\zeta\in\mathcal{Z}} \frac{1}{2\pi} \left(\sum_{i=1}^{n} h_i e^{\lambda_{\zeta,i} t} \boldsymbol{v}_i(\zeta, 0) \right) e^{j\zeta x} \right\| dx \end{aligned} \tag{A.2}$$

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$$\leq \frac{1}{2\pi} e^{-\gamma t} \int_{\Omega} \left\| \sum_{\zeta \in \mathcal{Z}} \left(\sum_{i=1}^{n} h_i \boldsymbol{v}_i(\zeta, 0) \right) e^{j\zeta x} \right\| dx$$
$$= \frac{1}{2\pi} e^{-\gamma t} \int_{\Omega} \| \boldsymbol{c}(x, 0) \| dx,$$

where the first equality is due to inverse Fourier transform. The right-hand side converges to zero as $t \to \infty$.

($\Leftarrow$) We prove by contraposition. Suppose there exists $\zeta = \zeta_0$ such that a root of $\varphi(\zeta_0, s) = 0$ does not lie in the left-half complex plane, that is, there exists $\hat{\lambda}$ such that $\varphi(\zeta_0, \hat{\lambda}) = 0$ and $\mathrm{Re}[\hat{\lambda}] \geq 0$. Then, (A.2) can be bounded from below by

$$\begin{aligned} \int_{\Omega} \left\| \frac{1}{2\pi} \sum_{\zeta \in \mathcal{Z}} \tilde{\boldsymbol{c}}(\zeta, t) e^{j\zeta x} \right\| dx &= \int_{\Omega} \left\| \sum_{\zeta \in \mathcal{Z}} \frac{1}{2\pi} \left(\sum_{i=1}^{n} h_i e^{\lambda_{\zeta,i} t} \boldsymbol{v}_i(\zeta, 0) \right) e^{j\zeta x} \right\| dx \\ &\geq \frac{1}{2\pi} e^{\hat{\lambda} t} \| h_i \boldsymbol{v}_i(\zeta_0, 0) \|. \end{aligned} \tag{A.3}$$

Since the right-hand side does not converge to zero in the limit of $t \to \infty$ unless $h_i \boldsymbol{v}_i(\zeta_0, 0) = 0$, the system is not asymptotically stable. $\square$

Proof of Proposition 1

($\Leftarrow$) Suppose there exists N_i such that $M_i + N_i \succeq 0$ and $\sum_{(j,k) \in \Theta_\ell} \nu_{jk}^{(i)} = 0$ for $\ell = 2, 3, \cdots 2\ell_i + 2$. It then follows that

$$\begin{aligned} \boldsymbol{z}_i^T (M_i + N_i) \boldsymbol{z}_i &= \boldsymbol{z}_i^T M_i \boldsymbol{z}_i + \boldsymbol{z}_i^T N_i \boldsymbol{z}_i \\ &= \boldsymbol{z}_i^T M_i \boldsymbol{z}_i + \sum_{j=1}^{\ell_i+1} \sum_{k=1}^{\ell_i+1} \nu_{jk}^{(i)} z^j z^k \\ &= \boldsymbol{z}_i^T M_i \boldsymbol{z}_i + \sum_{\ell=2}^{2\ell_i+2} \left(\sum_{(j,k) \in \Theta_\ell} \nu_{jk}^{(i)} \right) z^\ell = \boldsymbol{z}_i^T M_i \boldsymbol{z} = \Delta_i(\zeta). \end{aligned} \tag{A.4}$$

Thus, $M_i + N_i \succeq 0$ implies $\Delta_i(\zeta) \geq 0$ for all $\zeta \in \mathbb{R}$.

($\Rightarrow$) The proof is based on the known result in algebra that $\Delta_i(\zeta)$ can be decomposed into sums of squares when $\Delta_i(\zeta) \geq 0$. Here we show the proof for self-completeness of the document.

Suppose $\Delta_i(\zeta) \geq 0$. It is obvious that $\deg(\Delta_i(\zeta))$ is even and the leading coefficient is positive. Otherwise $\Delta_i(\zeta) < 0$ as $\zeta \to \infty$ or $\zeta \to -\infty$. Moreover, all roots of $\Delta_i(\zeta) = 0$ must be either pairs of complex conjugates and/or real numbers with even multiplicity because the sign of $\Delta_i(\zeta)$ alters at the real roots with odd multiplicity. Therefore, we can write

$$\begin{aligned}\Delta_i(\zeta) &= \prod_p \left(\zeta - (\sigma_p + j\gamma_p)\right)\left(\zeta - (\sigma_p - j\gamma_p)\right) \prod_q (\zeta - \eta_q)^{2r_q} \\ &= \prod_p \left((\zeta - \sigma_p)^2 + \gamma_p^2\right) \prod_q (\zeta - \eta_q)^{2r_q},\end{aligned} \tag{A.5}$$

where $\{\sigma_p \pm j\gamma_p\}_p$ and $\{\eta_q\}_q$ are the complex and real roots of $\Delta_i(\zeta) = 0$, respectively. The right-hand side of (A.5) shows that $\Delta_i(\zeta)$ is the sums of squares. Thus, it is possible to write

$$\Delta_i(\zeta) = \sum_j (\boldsymbol{z}_i^T \boldsymbol{\phi}_{ij})^2 = \boldsymbol{z}_i^T \left(\sum_j \boldsymbol{\phi}_{ij}\boldsymbol{\phi}_{ij}^T\right) \boldsymbol{z}_i = \boldsymbol{z}_i^T \Phi_i \boldsymbol{z}, \tag{A.6}$$

where $\boldsymbol{\phi}_{ij} := [\phi_{i0}, \phi_{i1}, \phi_{i2}, \cdots, \phi_{i\ell_i}]^T \in \mathbb{R}^{\ell_i+1}$ is the vector of the coefficients of the polynomials in the j-th square, and $\Phi_i := \sum_j \boldsymbol{\phi}_{ij}\boldsymbol{\phi}_{ij}^T$. Since $\Delta_i(\zeta)$ is also represented by $\Delta_i(\zeta) = \boldsymbol{z}_i^T M_i \boldsymbol{z}_i$, there exists a matrix N_i satisfying $\boldsymbol{z}_i^T N_i \boldsymbol{z}_i = 0$ and $\Phi_i = M_i + N_i$. Thus, it follows from the definition of Φ_i that $M_i + N_i \succeq 0$. Moreover, $\boldsymbol{z}_i^T N_i \boldsymbol{z}_i = 0$ implies $\sum_{(j,k)\in\Theta_\ell} \nu_{jk}^{(i)} = 0$ for $\ell = 2, 3, \cdots, 2\ell_i + 2$. □

Proof of Proposition 2

We first show the equivalence of (i) and (iii).

Case of $\mathcal{I} = [\underline{\zeta}, \overline{\zeta}]$

Define $\tilde{\zeta}$ by

$$\tilde{\zeta} := \frac{(\overline{\zeta} - \underline{\zeta})\zeta + (\overline{\zeta} + \underline{\zeta})}{2}. \tag{A.7}$$

By the change of variable, it follows that $\Delta_i(\tilde{\zeta}) > 0$ for all $\tilde{\zeta} \in \mathcal{I}$ is equivalent to $\Delta_i(\zeta) > 0$ for all $\zeta \in [-1, 1]$. According to Fekete (1935) (see Theorem 2.6 of [1] for example), $\Delta_i(\zeta)$ is non-negative for all $\zeta \in [-1, 1]$ if and only if there exist non-negative polynomials $f(\zeta)$

and $g(\zeta)$ satisfying

$$\Delta_i(\zeta) = f(\zeta) + (1 - \zeta^2)g(\zeta). \tag{A.8}$$

Since a univariate non-negative polynomial can always be decomposed into sums of squares (see the proof for Proposition 1), we can write $f(\zeta) := \boldsymbol{z}_1^T K_i \boldsymbol{z}_1$ and $g(\zeta) := \boldsymbol{z}_2^T L_i \boldsymbol{z}_2$ using positive semidefinite matrices K_i and L_i. The equality constraints are obtained by substituting these matrix representations into (A.8) and equating both sides of the equation.

Case of $\mathcal{I} = [\underline{\zeta}, \infty)$:
Let $\tilde{\zeta} := \zeta + \underline{\zeta}$. Then, $\Delta_i(\tilde{\zeta}) \geq 0$ for $\tilde{\zeta} \in [\underline{\zeta}, \infty)$ is equivalent to $\Delta_i(\zeta) \geq 0$ for $\zeta \in [0, \infty)$. The latter condition is equivalent to the existence of non-negative functions $f(\zeta)$ and $g(\zeta)$ such that

$$\Delta_i(\zeta) = f(\zeta) + \zeta g(\zeta) \tag{A.9}$$

(see Theorem 2.7 of [1]). This is equivalent to the existence of positive semidefinite matrices K_i and L_i, where $f(\zeta) := \boldsymbol{z}_1^T K_i \boldsymbol{z}_1$ and $g(\zeta) := \boldsymbol{z}_2^T L_i \boldsymbol{z}_2$. The proof is complete by substituting these into (A.9) and writing each entry of both sides of the equation.

Case of $\mathcal{I} = (-\infty, \overline{\zeta}]$:
Let $\tilde{\zeta} := -\zeta + \overline{\zeta}$. Then, $\Delta_i(\tilde{\zeta}) \geq 0$ for $\tilde{\zeta} \in (-\infty, \overline{\zeta}]$ is equivalent to $\Delta_i(\zeta) \geq 0$ for $\zeta \in [0, \infty)$. The rest of the proof is the same as the case of $\mathcal{I} = [\underline{\zeta}, \infty)$.

(ii) $\Leftrightarrow$ (iii): The results are obtained by changing the variable ζ in (A.8) and (A.9) with $\tilde{\zeta}$ defined above for each interval $\mathcal{I}$.

References

[1] Lasserre JB. Moments, Positive Polynomials and Their Applications. Imperial College Press; 2009.