This worksheet solves Taylor's ODE to determine the instantaneous geostrophic flow, both analytically where possible and numerically using a least squares scheme.
The Taylor equation is in general not valid due to an incorrect treatmment of the boundary conditions and so these solutions are not correct, except for the specific case of purely toroidal non-axisymmetric magnetic fields.

```
> restart:
> Digits := 50:
> with(orthopoly, P):
  with(VectorCalculus):
  SetCoordinates(cartesian[x,y,z]):
```

> # Define some useful routines to construct spherical harmonics

```
> # L1 is theta-factor in spherical harmonic (note that P(l,x) is
    the l-th Legendre polynomial)
> L1 := (l,m) -> if type(m, numeric) then if m <> 0 then sin(theta)
  ^abs(m) * subs(z=cos(theta), diff(P(l,z),z$abs(m))) else P(l, cos
  (theta)) end if else 'L1'(l,m) end if;
```

$$L1 := (l, m) \rightarrow \textbf{if } type(m, numeric) \textbf{ then if } m \neq 0 \textbf{ then } \sin(\theta)^{|m|} subs\left(z=\cos(\theta), \frac{\partial^{|m|}}{\partial z^{|m|}} P(l, z)\right) \textbf{ else } P(l, \cos(\theta)) \textbf{ end if else } 'L1'(l, m) \textbf{ end if} \quad \textbf{(1)}$$

```
> # Lp is phi-factor
> Lp := m -> if type(m, numeric) then if m = 0 then 1 elif m < 0
  then sin(-m*phi) else cos(m*phi) end if else 'Lp'(m) end if;
```

$$Lp := m \rightarrow \textbf{if } type(m, numeric) \textbf{ then if } m = 0 \textbf{ then } 1 \textbf{ elif } m < 0 \textbf{ then } \sin(VectorCalculus\text{:-}`\text{-}`(m\,\phi)) \textbf{ else } \cos(m\,\phi) \textbf{ end if else } 'Lp'(m) \textbf{ end if} \quad \textbf{(2)}$$

```
> # norm of L1*Lp is integral of (L1*Lp)^2 over sphere is int((L1*
  Lp)^2 * sin(theta), theta=0..Pi, phi=0..2*Pi)
> L2norm_squared := (l,m) -> int(L1(l,m)^2 * sin(theta), theta=0..
  Pi) * int(Lp(m)^2, phi=0..2*Pi) / (4*Pi);
```

$$L2norm\_squared := (l, m) \rightarrow VectorCalculus\text{:-}int\left(L1(l, m)^2 \sin(\theta), \theta = 0\,..\,\pi\right) VectorCalculus\text{:-}int\left(Lp(m)^2, \phi = 0\,..\,2\,\pi\right) \frac{1}{4\,\pi} \quad \textbf{(3)}$$

```
> # L2 is Schmidt quasi-normalised spherical harmonic
> L2 := (l,m) -> if type(l,numeric) and type(m,numeric) then L1(l,
  m) * Lp(m) / sqrt(L2norm_squared(l,m)) / sqrt(2*l+1) else 'L2'(l,
  m) end if;
```

$$L2 := (l, m) \rightarrow \textbf{if } type(l, numeric) \textbf{ and } type(m, numeric) \textbf{ then } L1(l, m)\, Lp(m) \frac{1}{\sqrt{L2norm\_squared(l, m)}} \frac{1}{\sqrt{2\,l+1}} \textbf{ else } 'L2'(l, m) \textbf{ end if} \quad \textbf{(4)}$$

```
> # Convert an expression in spherical coordinates to Cartesian
```

```
coordinates.
> sph2cart := proc(expr)
    local res;
   res := expand(expr, trig);
   res := subs(cos(phi) = x/(r*sin(theta)), sin(phi) = y/(r*sin
(theta)), res);
   res := subs(cos(theta) = z/r, sin(theta) = sqrt(x^2+y^2)/r,
res);
   res := subs(r = sqrt(x^2+y^2+z^2), res);
   return simplify(res)
end proc:
> # Construct vector field from poloidal and toroidal scalars
scalars2vf := proc(tor_scalar, pol_scalar)
   return simplify(Curl(VectorField(sph2cart(tor_scalar/r) * <x,
y,z>))
                    + Curl(Curl(VectorField(sph2cart(pol_scalar/r)
* <x,y,z>))))
end proc:
> # Define basis functions for the poloidal flow which vanish at r=
1. Degree is l+2n-1.
> Chi_n := (l,n) -> r^(l+1) * (1-r^2) * P(n-1,3/2,l+1/2,2*r^2-1): #
curl of this is basis for poloidal part
> W_n := (l,n) -> r^(l+1) * P(n,-1/2,l+1/2,2*r^2-1): # curl^2 of
this is basis for toroidal part
> Psi_n := proc (l, n) options operator, arrow; r^(l+1)*((-2*n^2*
(l+1)-n*(l+1)*(2*l-1)-l*(2*l+1))*P(n, 0, l+1/2, 2*r^2-1)+((2*l+2)
*n^2+(2*l+3)*(l+1)*n+(2*l+1)^2)*P(n-1, 0, l+1/2, 2*r^2-1)+4*n*l+
l*(2*l+1))end proc
```

$$Psi\_n := (l, n) \rightarrow r^{l+1} \Big(\big(VectorCalculus{:-}\text{`-`}(2\, n^2\, (l+1)) + VectorCalculus{:-}\text{`-`}(n\, (l+1)\, (2\, l + (-1))) + VectorCalculus{:-}\text{`-`}(l\, (2\, l+1)) \big)\, P\Big(n, 0, l+1\,\frac{1}{2}, 2\, r^2 + (-1)\Big) + \big((2\, l+2)\, n^2 + (2\, l+3)\, (l+1)\, n + (2\, l+1)^2 \big)\, P\Big(n + (-1), 0, l+1\,\frac{1}{2}, 2\, r^2 + (-1)\Big) + 4\, n\, l + l\, (2\, l+1) \Big) \qquad (5)$$

> # Define magnetic field

```
> # choose a number corresponding to the chosen magnetic field 1=
axisymmetric poloidal, 2=nonaxisymmetric toroidal, 3=
nonaxisymmetric poloidal, 4=nonaxisymmetric mixed state
```

> $k := 1;$

$$k := 1 \qquad (6)$$

> **if** $k = 1$ **then B_scalar_tor** $:=$ **0 : B_scalar_pol** $:=$ *eval*(**r^2 * (30 * r^4 − 57 * r^2 + 25) * L2(l, m), {l = 1, m = 0, n = 1})**
end if

$$B\_scalar\_tor := 0$$

$$B\_scalar\_pol := r^2\, (30\, r^4 - 57\, r^2 + 25)\, \cos(\theta) \qquad (7)$$

```
> if k = 2 then B_scalar_tor := simplify(eval(Chi_n(l, n)·L2(l, m), {l = 1, m = 1, n = 1})) :
      B_scalar_pol := 0
  end if
> if k = 3 then B_scalar_tor := 0 : B_scalar_pol := eval(Psi_n(l, n)·L2(l, m), {l = 2, m = 2, n
      = 1})
  end
> if k = 4 then B_scalar_tor := eval(Chi_n(l, n)·L2(l, m), {l = 2, m = 1, n = 1}) :
      B_scalar_pol := eval(Psi_n(l, n)·L2(l, m), {l = 2, m = 1, n = 1})
  end if
```

> $B\_cart\_pol := scalars2vf(0, B\_scalar\_pol);$

$$B\_cart\_pol := \left(-120\, x^3 z - 120\, x y^2 z - 120\, x z^3 + 114\, x z\right)\bar{e}_x + \left(-120\, x^2 y z - 120\, y^3 z - 120\, y z^3 + 114\, y z\right)\bar{e}_y + \left(180\, x^4 + 360\, x^2 y^2 + 240\, x^2 z^2 + 180\, y^4 + 240\, y^2 z^2 + 60\, z^4 - 228\, x^2 - 228\, y^2 - 114\, z^2 + 50\right)\bar{e}_z \qquad (8)$$

> $B\_cart\_tor := scalars2vf(B\_scalar\_tor, 0);$

$$B\_cart\_tor := 0\bar{e}_x \qquad (9)$$

> $B\_sph\_pol := simplify(MapToBasis(B\_cart\_pol, spherical[r, \text{theta}, \text{phi}]));$

$$B\_sph\_pol := \left(\left(60\, r^4 - 114\, r^2 + 50\right)\cos(\theta)\right)\bar{e}_r + \left(\left(-180\, r^4 + 228\, r^2 - 50\right)\sin(\theta)\right)\bar{e}_\theta \qquad (10)$$

> $B\_sph\_tor := simplify(MapToBasis(B\_cart\_tor, spherical[r, \text{theta}, \text{phi}]));$

$$B\_sph\_tor := 0\bar{e}_r \qquad (11)$$

> $SetCoordinates(spherical[r, \text{theta}, \text{phi}]);$

$$spherical_{r,\,\theta,\,\phi} \qquad (12)$$

> *#Scale the magnetic field*

> $Scale\_pol := \text{sqrt}\left(\frac{1}{4\cdot\text{Pi}}\cdot int\left(int\left(int\left(A\_pol\cdot B\_sph\_pol.B\_sph\_pol\cdot r^2\cdot\sin(\text{theta}), \text{phi} = 0..2\cdot\text{Pi}\right), \text{theta} = 0..\text{Pi}\right), r = 0..1\right)\right);$

$$Scale\_pol := \frac{2}{231}\sqrt{1188726}\,\sqrt{A\_pol} \qquad (13)$$

> **if** $Scale\_pol = 0$ **then** $A\_pol = 0$
> **else** $A\_pol := simplify(solve(Scale\_pol = 1, A\_pol));$ **end if**;

$$A\_pol := \frac{231}{20584} \qquad (14)$$

> $Scale\_tor := \text{sqrt}\left(\frac{1}{4\cdot\text{Pi}}\cdot int\left(int\left(int\left(A\_tor\cdot B\_sph\_tor.B\_sph\_tor\cdot r^2\cdot\sin(\text{theta}), \text{phi} = 0..2\right.\right.\right.\right.$

$\cdot$ Pi), theta = 0 .. Pi), $r = 0 .. 1$));

$$Scale\_tor := 0 \tag{15}$$

> **if** $Scale\_tor = 0$ **then** $A\_tor = 0$
> **else** $A\_tor := simplify(solve(Scale\_tor = 1, A\_tor))$ **end if**;

$$A\_tor = 0 \tag{16}$$

> $SetCoordinates(cartesian[x, y, z])$:

> $B\_cart := scalars2vf(\mathrm{sqrt}(A\_tor) \cdot B\_scalar\_tor, \mathrm{sqrt}(A\_pol) \cdot B\_scalar\_pol)$;

$$B\_cart := \left(-\frac{30}{2573}\sqrt{1188726}\,x\,y^2 z - \frac{30}{2573}\sqrt{1188726}\,x^3 z - \frac{30}{2573}\sqrt{1188726}\,x\,z^3 + \frac{57}{5146}\sqrt{1188726}\,x\,z\right)\bar{e}_x + \left(-\frac{30}{2573}\sqrt{1188726}\,x^2 y\,z - \frac{30}{2573}\sqrt{1188726}\,y^3 z - \frac{30}{2573}\sqrt{1188726}\,y\,z^3 + \frac{57}{5146}\sqrt{1188726}\,y\,z\right)\bar{e}_y + \left(\frac{90}{2573}\sqrt{1188726}\,x^2 y^2 + \frac{60}{2573}\sqrt{1188726}\,x^2 z^2 + \frac{60}{2573}\sqrt{1188726}\,y^2 z^2 + \frac{45}{2573}\sqrt{1188726}\,x^4 + \frac{45}{2573}\sqrt{1188726}\,y^4 + \frac{15}{2573}\sqrt{1188726}\,z^4 - \frac{57}{2573}\sqrt{1188726}\,x^2 - \frac{57}{2573}\sqrt{1188726}\,y^2 - \frac{57}{5146}\sqrt{1188726}\,z^2 + \frac{25}{5146}\sqrt{1188726}\right)\bar{e}_z \tag{17}$$

```
> B_sph := simplify(MapToBasis(B_cart, spherical[r,theta,phi]));
```

$$B\_sph := \frac{1}{5146}\cos(\theta)\sqrt{1188726}\,(30 r^4 - 57 r^2 + 25)\bar{e}_r - \frac{1}{5146}\sin(\theta)\sqrt{1188726}\,(90 r^4 - 114 r^2 + 25)\bar{e}_\theta \tag{18}$$

> # Compute rhs of magnetostrophic equation

```
> # slaved equation is Omega cross u = -div(p) + curl(B) cross B, we ignore the pressure
> RHS := CrossProduct(Curl(B_cart), B_cart): simplify(RHS);
```

$$-\frac{3465}{5146}\,x\,(28 x^2 + 28 y^2 + 28 z^2 - 19)\,(90 x^4 + 180 x^2 y^2 + 120 x^2 z^2 + 90 y^4 + 120 y^2 z^2 + 30 z^4 - 114 x^2 - 114 y^2 - 57 z^2 + 25)\bar{e}_x - \frac{3465}{5146}\,y\,(28 x^2 + 28 y^2 + 28 z^2 - 19)\,(90 x^4 \tag{19}$$

$$+180\,x^2y^2+120\,x^2z^2+90\,y^4+120\,y^2z^2+30\,z^4-114\,x^2-114\,y^2-57\,z^2+25)\bar{e}_y$$

$$-\frac{2910600}{2573}\,(x^2+y^2)\left(x^2+y^2+z^2-\frac{19}{28}\right)\left(x^2+y^2+z^2-\frac{19}{20}\right)z\bar{e}_z$$

```
> map(factor, simplify(MapToBasis(RHS, spherical[r,theta,phi])));
  # for comparison
```

$$-\frac{3465}{5146}\sin(\theta)^2\,(28\,r^2-19)\,(90\,r^4-114\,r^2+25)\,r\bar{e}_r-\frac{3465}{5146}\cos(\theta)\,r\sin(\theta)\,(28\,r^2-19)\,(30\,r^4-57\,r^2+25)\bar{e}_\theta \tag{20}$$

> # Construct basis for u

```
> # Let N be the degree of B. Then the degree of curl(B) cross B is
  (N-1) + N = 2N-1. Thus the degree of u is also 2N-1. Thus, the
  poloidal scalar has degree 2N+1 (we have to undo two curls) and
  the toroidal scalar has degree 2N. Furthermore, the m-degree of
  curl(B) cross B is twice the m-degree of B.
> B_degree := max(seq(degree(B_cart[idx], {x,y,z}), idx = 1 .. 3));
```

$$B\_degree := 4 \tag{21}$$

```
> Max_pol_degree := 2 * B_degree + 1; Max_tor_degree := 2 *
  B_degree;
```

$$Max\_pol\_degree := 9$$

$$Max\_tor\_degree := 8 \tag{22}$$

```
> Max_m_degree := 2 * max(seq(degree(expand(B_sph[idx], trig), {cos
  (phi), sin(phi)}), idx = 1 .. 3));
```

$$Max\_m\_degree := 0 \tag{23}$$

```
> # For the particular example here, certain modes are zero, but we
  will not exploit this knowledge.
> # Also note that unless B is a Taylor state, there will be no
  solution to the magnetostrophic equation.

> u_scalar_pol := add(add(add(S[l,m,n] * L2(l,m) * Chi_n(l,n),
                          m = -min(l, Max_m_degree) .. min(l,
  Max_m_degree)),
                           n = 1 .. (Max_pol_degree - l + 1) /
  2),
                           l = 1 .. Max_pol_degree):
> u_scalar_tor := add(add(add(T[l,m,n] * L2(l,m) * W_n(l,n),
                          m = -min(l, Max_m_degree) .. min(l,
  Max_m_degree)),
                          n = 0 .. floor((Max_tor_degree - l +
  1)/2)),
                           l = 1 .. Max_tor_degree):
> coeff(u_scalar_pol, S[2,-2,2]);
```

$$0 \tag{24}$$

```
> # u is poloidal part (curl^2 of scalar times hat r) + toroidal
  part (curl of scalar times hat r)
> u_cart := scalars2vf(u_scalar_tor, u_scalar_pol):
> variables := indets(u_cart, indexed);
```

$$variables := \{S_{1,0,1}, S_{1,0,2}, S_{1,0,3}, S_{1,0,4}, S_{2,0,1}, S_{2,0,2}, S_{2,0,3}, S_{2,0,4}, S_{3,0,1}, S_{3,0,2}, S_{3,0,3}, \tag{25}$$

$S_{4,0,1}, S_{4,0,2}, S_{4,0,3}, S_{5,0,1}, S_{5,0,2}, S_{6,0,1}, S_{6,0,2}, S_{7,0,1}, S_{8,0,1}, T_{1,0,0}, T_{1,0,1}, T_{1,0,2}, T_{1,0,3}, T_{1,0,4}, T_{2,0,0}, T_{2,0,1}, T_{2,0,2}, T_{2,0,3}, T_{3,0,0}, T_{3,0,1}, T_{3,0,2}, T_{3,0,3}, T_{4,0,0}, T_{4,0,1}, T_{4,0,2}, T_{5,0,0}, T_{5,0,1}, T_{5,0,2}, T_{6,0,0}, T_{6,0,1}, T_{7,0,0}, T_{7,0,1}, T_{8,0,0}\}$

> # Compute lhs of magnetostrophic equation

```
> # slaved equation is Omega cross u = -div(p) + curl(B) cross B
> Omega_vec := VectorField([0,0,1]);  # rotation vector in
  cartesian coordinates
```

$$Omega\_vec := \bar{e}_z \tag{26}$$

```
> LHS := CrossProduct(Omega_vec, u_cart):
```

> # Solve magnetostrophic equation for basis coefficients

```
> # take the curl of slaved equation; pressure drops out
> eqn := simplify(Curl(LHS - RHS)):
> constraints := `union`(seq({coeffs(collect(eqn[i], [x,y,z],
  distributed), [x,y,z])}, i=1..3)):
> nops(constraints); nops(variables);
```

$$113$$
$$44 \tag{27}$$

```
> sol1 := solve(constraints, variables);
```

$$sol1 := \Big\{S_{1,0,1}=0, S_{1,0,2}=0, S_{1,0,3}=0, S_{1,0,4}=0, S_{2,0,1}=0, S_{2,0,2}=0, S_{2,0,3}=0, S_{2,0,4}=0, S_{3,0,1}=0, S_{3,0,2}=0, S_{3,0,3}=0, S_{4,0,1}=0, S_{4,0,2}=0, S_{4,0,3}=0, S_{5,0,1}=0, S_{5,0,2}=0, S_{6,0,1}=0, S_{6,0,2}=0, S_{7,0,1}=0, S_{8,0,1}=0, T_{1,0,0}=T_{1,0,0}, T_{1,0,1}=T_{1,0,1}, T_{1,0,2}=T_{1,0,2}, T_{1,0,3}=T_{1,0,3}, T_{1,0,4}=0, T_{2,0,0}=0, T_{2,0,1}=0, T_{2,0,2}=0, T_{2,0,3}=0, T_{3,0,0}=\frac{27489}{10292}-\frac{1}{2}T_{1,0,1}, T_{3,0,1}=-\frac{23177}{2573}-\frac{14}{27}T_{1,0,2}, T_{3,0,2}=\frac{7350}{2573}-\frac{35}{66}T_{1,0,3}, T_{3,0,3}=0, T_{4,0,0}=0, T_{4,0,1}=0, T_{4,0,2}=0, T_{5,0,0}=\frac{16555}{2573}+\frac{10}{27}T_{1,0,2}, T_{5,0,1}=-\frac{69300}{33449}+\frac{5}{13}T_{1,0,3}, T_{5,0,2}=0, T_{6,0,0}=0, T_{6,0,1}=0, T_{7,0,0}=\frac{55125}{33449}-\frac{175}{572}T_{1,0,3}, T_{7,0,1}=0, T_{8,0,0}=0\Big\} \tag{28}$$

> # set geostrophic component to zero to remove degeneracy

```
> u_soln := simplify(subs(sol1, u_cart));
```

$$
\begin{aligned}
u\_soln :=\ & -\frac{3675}{64} \Bigg(\bigg(\Big(-\frac{5412}{2573} + T_{1,0,3} \Big) y^6 + \bigg(\Big(-\frac{16236}{2573} + 3\, T_{1,0,3} \Big) x^2 + \frac{55176}{12865} \\
& - \frac{12}{7}\, T_{1,0,3} + \frac{16}{63}\, T_{1,0,2} + \frac{25344}{2573}\, z^2 \bigg) y^4 + \bigg(\Big(-\frac{16236}{2573} + 3\, T_{1,0,3} \Big) x^4 + \Big(\frac{50688}{2573}\, z^2 \\
& + \frac{32}{63}\, T_{1,0,2} - \frac{24}{7}\, T_{1,0,3} + \frac{110352}{12865} \Big) x^2 - \frac{32}{105}\, T_{1,0,2} + \frac{6}{7}\, T_{1,0,3} - \frac{5280}{2573} \\
& - \frac{240768}{12865}\, z^2 + \frac{16}{245}\, T_{1,0,1} + \frac{25344}{2573}\, z^4 \bigg) y^2 + \Big(-\frac{5412}{2573} + T_{1,0,3} \Big) x^6 + \Big(\frac{55176}{12865} \\
& - \frac{12}{7}\, T_{1,0,3} + \frac{16}{63}\, T_{1,0,2} + \frac{25344}{2573}\, z^2 \Big) x^4 + \Big(-\frac{32}{105}\, T_{1,0,2} + \frac{6}{7}\, T_{1,0,3} - \frac{5280}{2573} \\
& - \frac{240768}{12865}\, z^2 + \frac{16}{245}\, T_{1,0,1} + \frac{25344}{2573}\, z^4 \Big) x^2 - \frac{32}{735}\, T_{1,0,1} + \frac{8}{105}\, T_{1,0,2} - \frac{4}{35}\, T_{1,0,3} \\
& + \frac{64}{3675}\, T_{1,0,0} + \frac{21120}{2573}\, z^2 - \frac{120384}{12865}\, z^4 + \frac{8448}{2573}\, z^6 \bigg) y \bar{e}_x + \frac{3675}{64} \bigg(\Big(-\frac{5412}{2573} \\
& + T_{1,0,3} \Big) x^6 + \bigg(\Big(-\frac{16236}{2573} + 3\, T_{1,0,3} \Big) y^2 + \frac{55176}{12865} - \frac{12}{7}\, T_{1,0,3} + \frac{16}{63}\, T_{1,0,2} \\
& + \frac{25344}{2573}\, z^2 \bigg) x^4 + \bigg(\Big(-\frac{16236}{2573} + 3\, T_{1,0,3} \Big) y^4 + \Big(\frac{50688}{2573}\, z^2 + \frac{32}{63}\, T_{1,0,2} - \frac{24}{7}\, T_{1,0,3} \\
& + \frac{110352}{12865} \Big) y^2 - \frac{32}{105}\, T_{1,0,2} + \frac{6}{7}\, T_{1,0,3} - \frac{5280}{2573} - \frac{240768}{12865}\, z^2 + \frac{16}{245}\, T_{1,0,1} \\
& + \frac{25344}{2573}\, z^4 \bigg) x^2 + \Big(-\frac{5412}{2573} + T_{1,0,3} \Big) y^6 + \Big(\frac{55176}{12865} - \frac{12}{7}\, T_{1,0,3} + \frac{16}{63}\, T_{1,0,2} \\
& + \frac{25344}{2573}\, z^2 \Big) y^4 + \Big(-\frac{32}{105}\, T_{1,0,2} + \frac{6}{7}\, T_{1,0,3} - \frac{5280}{2573} - \frac{240768}{12865}\, z^2 + \frac{16}{245}\, T_{1,0,1} \\
& + \frac{25344}{2573}\, z^4 \Big) y^2 - \frac{32}{735}\, T_{1,0,1} + \frac{8}{105}\, T_{1,0,2} - \frac{4}{35}\, T_{1,0,3} + \frac{64}{3675}\, T_{1,0,0} + \frac{21120}{2573}\, z^2 \\
& - \frac{120384}{12865}\, z^4 + \frac{8448}{2573}\, z^6 \bigg) x \bar{e}_y
\end{aligned}
\tag{29}
$$

```
> u_cyl := simplify(MapToBasis(u_soln, cylindrical[s,phi,z]));
```

$$
\begin{aligned}
u\_cyl :=\ & \frac{3675}{64} \bigg(\Big(-\frac{5412}{2573} + T_{1,0,3} \Big) s^6 + \Big(\frac{55176}{12865} - \frac{12}{7}\, T_{1,0,3} + \frac{16}{63}\, T_{1,0,2} \\
& + \frac{25344}{2573}\, z^2 \Big) s^4 + \Big(-\frac{32}{105}\, T_{1,0,2} + \frac{6}{7}\, T_{1,0,3} - \frac{5280}{2573} - \frac{240768}{12865}\, z^2 + \frac{16}{245}\, T_{1,0,1} \\
& + \frac{25344}{2573}\, z^4 \Big) s^2 + \frac{8448}{2573}\, z^6 - \frac{120384}{12865}\, z^4 + \frac{21120}{2573}\, z^2 + \frac{64}{3675}\, T_{1,0,0} - \frac{32}{735}\, T_{1,0,1} \\
& + \frac{8}{105}\, T_{1,0,2} - \frac{4}{35}\, T_{1,0,3} \bigg) s \bar{e}_{\phi}
\end{aligned}
\tag{30}
$$

```
> SetCoordinates(cylindrical[s, phi, z])
```

$$cylindrical_{s,\,\phi,\,z} \tag{31}$$

```
> geostrophic_component := simplify((int(int(u_cyl[2], phi = 0 ..
  2*Pi), z = -sqrt(-s^2+1) .. sqrt(-s^2+1)))/(4*Pi*sqrt(-s^2+1)))
```

$$
geostrophic\_component := \frac{3675}{64} \bigg(\Big(T_{1,0,3} - \frac{349932}{90055} \Big) s^6 + \Big(\frac{16}{63}\, T_{1,0,2} - \frac{12}{7}\, T_{1,0,3} \tag{32}
$$

$$+\frac{4235352}{450275}\Big)s^4+\Big(\frac{16}{245}T_{1,0,1}-\frac{32}{105}T_{1,0,2}+\frac{6}{7}T_{1,0,3}-\frac{3026144}{450275}\Big)s^2$$
$$+\frac{64}{3675}T_{1,0,0}-\frac{32}{735}T_{1,0,1}+\frac{8}{105}T_{1,0,2}-\frac{4}{35}T_{1,0,3}+\frac{600512}{450275}\Big)s$$

```
> u_cyl := simplify(u_cyl - VectorField([0, geostrophic_component, 0])); #remove geostrophic component from u_cyl
```

$$u\_cyl := \frac{231}{2573}s\left(1140s^6+6300s^4z^2+6300s^2z^4+2100z^6-3273s^4-11970s^2z^2-5985z^4+2986s^2+5250z^2-853\right)\bar{e}_\phi \tag{33}$$

```
> simplify(int(int( u_cyl[2],z=-sqrt(1-s^2)..sqrt(1-s^2)),phi=0..2*Pi)); #check that no geostrophic component remains
```

$$0 \tag{34}$$

```
> map(factor, simplify(MapToBasis(u_soln, spherical[r,theta,phi])))
; # added
```

$$-\frac{1}{494016}r\sin(\theta)\Big(8645280r^2T_{1,0,2}-24314850r^2T_{1,0,3}-7204400r^4T_{1,0,2}+48629700r^4T_{1,0,3}+3241980T_{1,0,3}-2161320T_{1,0,2}+1235040T_{1,0,1}-494016T_{1,0,0}-1852560r^2T_{1,0,1}+28367325\cos(\theta)^6r^6T_{1,0,3}-85101975\cos(\theta)^4r^6T_{1,0,3}-7204400\cos(\theta)^4r^4T_{1,0,2}+48629700\cos(\theta)^4r^4T_{1,0,3}+85101975\cos(\theta)^2r^6T_{1,0,3}+14408800\cos(\theta)^2r^4T_{1,0,2}-97259400\cos(\theta)^2r^4T_{1,0,3}+1852560\cos(\theta)^2r^2T_{1,0,1}-8645280\cos(\theta)^2r^2T_{1,0,2}+24314850\cos(\theta)^2r^2T_{1,0,3}-28367325r^6T_{1,0,3}+59667300r^6-291060000r^2\cos(\theta)^2-387109800r^4\cos(\theta)^4-152806500r^6\cos(\theta)^6+458419500r^6\cos(\theta)^4-458419500r^6\cos(\theta)^2+774219600r^4\cos(\theta)^2-121663080r^4+58212000r^2\Big)\bar{e}_\phi \tag{35}$$

> # Construct the ODE for geostrophic flow

```
> B_cyl := simplify(MapToBasis(B_cart, cylindrical[s,phi,z]));
```

$$B\_cyl := -\frac{30}{2573}\sqrt{1188726}\left(s^2+z^2-\frac{19}{20}\right)z\,s\bar{e}_s+\frac{1}{5146}\left(90s^4+\left(120z^2-114\right)s^2+30z^4-57z^2+25\right)\sqrt{1188726}\,\bar{e}_z \tag{36}$$

> $iint(int(CrossProduct(s\cdot Curl(B\_cyl), B\_cyl)[2], z=-\mathrm{sqrt}(1-s\char`^2)..\mathrm{sqrt}(1-s\char`^2)), \mathrm{phi}=0..2\cdot\mathrm{Pi})$; *#check that it is taylor state*

$$0 \tag{37}$$

> $u\_geo := ug(s) \cdot VectorField([0, 1, 0]);$ *#introduce geostrophic component*

$$u\_geo := (ug(s))\bar{e}_{\phi} \tag{38}$$

> $B\_cyl\_dot := eval(Curl(CrossProduct(u\_cyl, B\_cyl)) + Curl(CrossProduct(u\_geo, B\_cyl)) + \mathrm{eta} \cdot Laplacian(B\_cyl));$ *# use induction equation*

$$B\_cyl\_dot := \Bigg(\eta\Bigg(-\frac{1}{s^2}\Bigg(-\frac{60}{2573}\sqrt{1188726}\left(s^2+z^2-\frac{19}{20}\right)z\,s-\frac{60}{2573}s^3\sqrt{1188726}\,z \tag{39}$$
$$+\frac{1}{5146}s\left(240 s^2 z+120 z^3-114 z\right)\sqrt{1188726}\Bigg)+\frac{1}{s}\Bigg(-\frac{60}{2573}\sqrt{1188726}\,s^2 z$$
$$-\frac{60}{2573}\sqrt{1188726}\left(s^2+z^2-\frac{19}{20}\right)z+\frac{1}{5146}\left(240 s^2 z+120 z^3\right.$$
$$\left.-114 z\right)\sqrt{1188726}\Bigg)-\frac{420}{2573}\sqrt{1188726}\,z\,s\Bigg)\Bigg)\bar{e}_s+\Bigg(\frac{231}{13240658}s\left(12600 s^4 z\right.$$
$$\left.+25200 s^2 z^3+12600 z^5-23940 s^2 z-23940 z^3+10500 z\right)\left(90 s^4+\left(120 z^2-114\right)s^2\right.$$
$$\left.+30 z^4-57 z^2+25\right)\sqrt{1188726}+\frac{231}{13240658}s\left(1140 s^6+6300 s^4 z^2+6300 s^2 z^4\right.$$
$$\left.+2100 z^6-3273 s^4-11970 s^2 z^2-5985 z^4+2986 s^2+5250 z^2-853\right)\left(240 s^2 z\right.$$
$$\left.+120 z^3-114 z\right)\sqrt{1188726}-\frac{13860}{6620329}s\left(1140 s^6+6300 s^4 z^2+6300 s^2 z^4+2100 z^6\right.$$
$$\left.-3273 s^4-11970 s^2 z^2-5985 z^4+2986 s^2+5250 z^2-853\right)\sqrt{1188726}\left(s^2+z^2\right.$$
$$\left.-\frac{19}{20}\right)z-\frac{6930}{6620329}s^2\left(6840 s^5+25200 s^3 z^2+12600 s z^4-13092 s^3-23940 s z^2\right.$$
$$\left.+5972 s\right)\sqrt{1188726}\left(s^2+z^2-\frac{19}{20}\right)z-\frac{13860}{6620329}s^3\left(1140 s^6+6300 s^4 z^2\right.$$
$$\left.+6300 s^2 z^4+2100 z^6-3273 s^4-11970 s^2 z^2-5985 z^4+2986 s^2+5250 z^2-853\right)$$
$$\sqrt{1188726}\,z+\frac{1}{5146}ug(s)\left(240 s^2 z+120 z^3-114 z\right)\sqrt{1188726}$$
$$-\frac{30}{2573}\left(\frac{\mathrm{d}}{\mathrm{d}s}ug(s)\right)\sqrt{1188726}\left(s^2+z^2-\frac{19}{20}\right)z\,s-\frac{60}{2573}ug(s)\sqrt{1188726}\,s^2 z$$
$$-\frac{30}{2573}ug(s)\sqrt{1188726}\left(s^2+z^2-\frac{19}{20}\right)z\Bigg)\bar{e}_{\phi}+\Bigg(\eta\Bigg(\frac{1}{s}\Bigg(-\frac{120}{2573}\sqrt{1188726}\,z^2 s$$

$$- \frac{60}{2573} \sqrt{1188726} \left(s^2 + z^2 - \frac{19}{20}\right) s - \frac{60}{2573} s^3 \sqrt{1188726} + \frac{1}{5146} s \left(240 s^2 + 360 z^2 - 114\right) \sqrt{1188726}\Bigg) - \frac{1}{s} \Bigg(- \frac{60}{2573} \sqrt{1188726} z^2 s - \frac{30}{2573} \sqrt{1188726} \left(s^2 + z^2 - \frac{19}{20}\right) s - \frac{1}{5146} \left(360 s^3 + 2 \left(120 z^2 - 114\right) s\right) \sqrt{1188726} + s \left(- \frac{60}{2573} \sqrt{1188726} z^2 - \frac{60}{2573} \sqrt{1188726} s^2 - \frac{30}{2573} \sqrt{1188726} \left(s^2 + z^2 - \frac{19}{20}\right) - \frac{1}{5146} \left(1080 s^2 + 240 z^2 - 228\right) \sqrt{1188726}\right)\Bigg)\Bigg)\bar{e}_z$$

> *#For Taylor ODE - attempt to solve analytically, only possible for a few specific magnetic fields*

> *# Taylor's ODE is alpha * diff(ug(s)/s,s,s) + beta * diff(ug(s)/s,s) = G, So we calulate all the coefficients, so we can formulate the ODE*

```
> SetCoordinates(cylindrical[s,phi,z]);
```

$$cylindrical_{s, \phi, z} \tag{40}$$

```
> alpha_integrand := s^2 * B_cyl[1]^2;
```

$$alpha\_integrand := \frac{415800}{2573} s^4 \left(s^2 + z^2 - \frac{19}{20}\right)^2 z^2 \tag{41}$$

```
> alpha := simplify(int(int(alpha_integrand, phi=0..2*Pi), z=-sqrt(1-s^2)..sqrt(1-s^2)));
```

$$\alpha := -\frac{198}{2573} s^4 \pi \sqrt{-s^2 + 1} \left(640 s^6 - 1808 s^4 + 1703 s^2 - 535\right) \tag{42}$$

```
> beta_integrand := simplify( s* (2 * B_cyl[1]^2 + s * DotProduct(B_cyl, Gradient(B_cyl[1]))));
```

$$beta\_integrand := -\frac{623700}{2573} \left(s^6 + \left(z^2 - \frac{133}{60}\right) s^4 + \left(-z^4 - \frac{19}{30} z^2 + \frac{1333}{900}\right) s^2 - z^6 + \frac{19}{12} z^4 - \frac{37}{100} z^2 - \frac{19}{72}\right) s^3 \tag{43}$$

```
> beta := simplify(int(int(beta_integrand, phi=0..2*Pi), z=-sqrt(1-s^2)..sqrt(1-s^2)));
```

$$\beta := -\frac{198}{2573} s^3 \pi \sqrt{-s^2 + 1} \left(7680 s^6 - 17440 s^4 + 12456 s^2 - 2689\right) \tag{44}$$

```
> G_integrand1 := simplify(CrossProduct(Curl(Curl(CrossProduct(u_cyl, B_cyl))), B_cyl)[2]);
```

$$G\_integrand1 := \frac{1737540882000}{6620329} \left(s^{12} + \left(\frac{476}{201} z^2 - \frac{4628}{1005}\right) s^{10} + \left(\frac{1189}{201} z^4 - \frac{4529}{402} z^2 \right.\right. \tag{45}$$

$$+\frac{512113}{60300}\bigg)s^8+\bigg(\frac{2384}{201}z^6-\frac{1662}{67}z^4+\frac{923588}{45225}z^2-\frac{2408299}{301500}\bigg)s^6+\bigg(-\frac{1284091}{72360}z^2+\frac{21801737}{5427000}+\frac{2275}{201}z^8-\frac{33863}{1005}z^6+\frac{3261827}{90450}z^4\bigg)s^4+\bigg(-\frac{109136}{5025}z^4-\frac{27626}{27135}+\frac{980}{201}z^{10}-\frac{1330}{67}z^8+\frac{54854}{1809}z^6+\frac{1667068}{226125}z^2\bigg)s^2+\frac{175}{201}\bigg(z^4-\frac{19}{10}z^2+\frac{5}{6}\bigg)^2\bigg(z^4-\frac{57}{50}z^2+\frac{1}{6}\bigg)\bigg)s$$

```
> G_integrand2 := simplify(CrossProduct(Curl(B_cyl), Curl(CrossProduct(u_cyl, B_cyl)))[2]);
```

$$G\_integrand2 := 0 \quad (46)$$

```
> G_integrand3 := simplify(CrossProduct(Curl(Laplacian(B_cyl)), B_cyl)[2]);
```

$$G\_integrand3 := 0 \quad (47)$$

```
> G_integrand4 := simplify(CrossProduct(Curl(B_cyl), Laplacian(B_cyl))[2]);
```

$$G\_integrand4 := 0 \quad (48)$$

```
> for idx from 1 to 4 do G||idx := simplify(int(int(-G_integrand||idx*s, phi=0..2*Pi), z=-sqrt(1-s^2)..sqrt(1-s^2))) end do;
```

$$G1 := -\frac{66528}{86064277}s^2\pi\sqrt{-s^2+1}\,(788582400 s^{12}-3424588800 s^{10}+5871698056 s^8-5008127804 s^6+2183964721 s^4-440479243 s^2+28950670)$$

$$G2 := 0$$

$$G3 := 0$$

$$G4 := 0 \quad (49)$$

```
> G := add(G||idx, idx=1..4);
```

$$G := -\frac{66528}{86064277}s^2\pi\sqrt{-s^2+1}\,(788582400 s^{12}-3424588800 s^{10}+5871698056 s^8-5008127804 s^6+2183964721 s^4-440479243 s^2+28950670) \quad (50)$$

```
> taylor_ode := simplify(alpha * diff(ug(s)/s,s,s) + beta * diff(ug(s)/s,s) - G);
```

$$taylor\_ode := -\frac{126720}{2573}\sqrt{-s^2+1}\,\bigg(\bigg(10 s^7-\frac{108}{5}s^5+\frac{905}{64}s^3-\frac{1619}{640}s\bigg)\bigg(\frac{d}{ds}ug(s)\bigg)+\bigg(-10 s^6+\frac{108}{5}s^4-\frac{905}{64}s^2+\frac{1619}{640}\bigg)ug(s)+\frac{1}{21407360}s\,(s^2-1)\bigg(-264963686400 s^{10}+885698150400 s^8+21407360 s^5\bigg(\frac{d^2}{ds^2}ug(s)\bigg)-1087192396416 s^6-39068432 s^3\bigg(\frac{d^2}{ds^2}ug(s)\bigg)+595538545728 s^4+17895215 s\bigg(\frac{d^2}{ds^2}ug(s)\bigg) \quad (51)$$

$$- 138273600528\, s^2 + 9727425120 \Big) \Big) \pi\, s$$

> *#Solve Taylor's ODE for the geostrophic flow*

```
> sol_taylor_ode := simplify(dsolve(taylor_ode)):
```

> $sol\_taylor\_ode2 := simplify(subs(\_C1 = 0, sol\_taylor\_ode)):$

> *#Combine magnetostrophic and geostrophic flow and set angular momentum to zero*

```
> u_total_cyl_t := simplify(u_cyl[2] + rhs(sol_taylor_ode2)):
```

> *#calculate constant so the angular momentum is zero*

```
> angular_momentum_t := int(int(int(u_total_cyl_t * s, phi = 0 ..
  2*Pi), z = -sqrt(1-s^2) .. sqrt(1-s^2)), s = 0 .. 1):
>
```

```
> C_val_t := solve(Quadrature(int(int(u_total_cyl_t*s, phi = 0 ..
  2*Pi), z = -sqrt(-s^2+1) .. sqrt(-s^2+1)),s=0..1,method=gaussian
  [50]),_C2):
```

> $u\_res\_t := subs(\_C2 = C\_val\_t, sol\_taylor\_ode2):$

> $plot(rhs(u\_res\_t), s = 0 ..1,):$

```
Warning, expecting only range variable s in expression -1/33449*
s*(-336*Int((640*s^4-1168*s^2+535)^(264/535)*(788582400*s^10
-2636006400*s^8+3235691656*s^6-1772436148*s^4+411528573*s^2
-28950670)*s^(1619/535)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2
-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)*Int(1/s^
(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*(
(I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^
(1/2)),s)+336*Int((640*s^4-1168*s^2+535)^(264/535)*(788582400*
s^10-2636006400*s^8+3235691656*s^6-1772436148*s^4+411528573*s^2
-28950670)*s^(1619/535)*Int(1/s^(2689/535)/(640*s^4-1168*
s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*
21^(1/2)-80*s^2+73))^(-304/1605*I*21^(1/2)),s)/(1/(s^2-1))^(1/2)
*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*I*21^
(1/2)),s)-33449*RootOf(Quadrature(277200/2573*s^2*Pi*(-s^2+1)^
(7/2)+1164240/2573*(-s^2+1)^(5/2)*Pi*s^4-1106028/2573*s^2*Pi*(-
s^2+1)^(5/2)+1940400/2573*(-s^2+1)^(3/2)*Pi*s^6-3686760/2573*(-
s^2+1)^(3/2)*Pi*s^4+1617000/2573*s^2*Pi*(-s^2+1)^(3/2)
+1053360/2573*s^8*Pi*(-s^2+1)^(1/2)-3024252/2573*s^6*Pi*(-s^2+1)
^(1/2)+2759064/2573*s^4*Pi*(-s^2+1)^(1/2)-38909700480/33449*s^2*
Pi*(-s^2+1)^(1/2)*Int(1/s^(2689/535)/(640*s^4-1168*s^2+535)^
(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)
-80*s^2+73))^(-304/1605*I*21^(1/2)),s)*Int((640*s^4-1168*
s^2+535)^(264/535)*s^(1619/535)/(1/(s^2-1))^(1/2)*((I*21^(1/2)
+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)
+553094402112/33449*s^2*Pi*(-s^2+1)^(1/2)*Int(1/s^(2689/535)/
(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)
+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^(1/2)),s)*
Int((640*s^4-1168*s^2+535)^(264/535)*s^(2689/535)/(1/(s^2-1))^
(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*
I*21^(1/2)),s)-2382154182912/33449*s^2*Pi*(-s^2+1)^(1/2)*Int
```

```
(1/s^(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^
(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^
(-304/1605*I*21^(1/2)),s)*Int((640*s^4-1168*s^2+535)^(264/535)*
s^(3759/535)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^
(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)+4348769585664/33449*
s^2*Pi*(-s^2+1)^(1/2)*Int(1/s^(2689/535)/(640*s^4-1168*s^2+535)^
(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)
-80*s^2+73))^(-304/1605*I*21^(1/2)),s)*Int((640*s^4-1168*
s^2+535)^(264/535)*s^(4829/535)/(1/(s^2-1))^(1/2)*((I*21^(1/2)
+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)
-3542792601600/33449*s^2*Pi*(-s^2+1)^(1/2)*Int(1/s^(2689/535)/
(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)
+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^(1/2)),s)*
Int((640*s^4-1168*s^2+535)^(264/535)*s^(5899/535)/(1/(s^2-1))^
(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*
I*21^(1/2)),s)+1059854745600/33449*s^2*Pi*(-s^2+1)^(1/2)*Int
(1/s^(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^
(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^
(-304/1605*I*21^(1/2)),s)*Int((640*s^4-1168*s^2+535)^(264/535)*
s^(6969/535)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^
(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)+4*s^2*Pi*(-s^2+1)^
(1/2)*_Z+38909700480/33449*s^2*Pi*(-s^2+1)^(1/2)*Int((640*s^4
-1168*s^2+535)^(264/535)*s^(1619/535)*Int(1/s^(2689/535)/(640*
s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*
s^2-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^(1/2)),s)/(1/
(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^
(304/1605*I*21^(1/2)),s)-553094402112/33449*s^2*Pi*(-s^2+1)^
(1/2)*Int((640*s^4-1168*s^2+535)^(264/535)*s^(2689/535)*Int(1/s^
(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*(
(I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^
(1/2)),s)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)
-80*s^2+73))^(304/1605*I*21^(1/2)),s)+2382154182912/33449*s^2*
Pi*(-s^2+1)^(1/2)*Int((640*s^4-1168*s^2+535)^(264/535)*s^
(3759/535)*Int(1/s^(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*
(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73)
)^(-304/1605*I*21^(1/2)),s)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*
s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)
-4348769585664/33449*s^2*Pi*(-s^2+1)^(1/2)*Int((640*s^4-1168*
s^2+535)^(264/535)*s^(4829/535)*Int(1/s^(2689/535)/(640*s^4
-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2
-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^(1/2)),s)/(1/(s^2
-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^
(304/1605*I*21^(1/2)),s)+3542792601600/33449*s^2*Pi*(-s^2+1)^
(1/2)*Int((640*s^4-1168*s^2+535)^(264/535)*s^(5899/535)*Int(1/s^
(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*(1/(s^2-1))^(1/2)*(
(I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73))^(-304/1605*I*21^
(1/2)),s)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)
-80*s^2+73))^(304/1605*I*21^(1/2)),s)-1059854745600/33449*s^2*
Pi*(-s^2+1)^(1/2)*Int((640*s^4-1168*s^2+535)^(264/535)*s^
(6969/535)*Int(1/s^(2689/535)/(640*s^4-1168*s^2+535)^(799/535)*
(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*s^2-73)/(I*21^(1/2)-80*s^2+73)
)^(-304/1605*I*21^(1/2)),s)/(1/(s^2-1))^(1/2)*((I*21^(1/2)+80*
s^2-73)/(I*21^(1/2)-80*s^2+73))^(304/1605*I*21^(1/2)),s)
-788172/2573*s^2*Pi*(-s^2+1)^(1/2),s = 0 .. 1,method = gaussian
[50]))) to be plotted but found names [method, Quadrature,
gaussian[50]]
```

> #numerical solution For Taylor's ODE

```
> N := 50; # order of chebysev expansion
```

$$N := 50 \quad (52)$$

```
> M := N + 1 : # order of expansion including logarithmic term
> with(orthopoly) :

> Taylor_chebsum := add(a_t||n·s·T(n, 2 s^2 − 1), n = 1 ..N) :

> Taylor_numerical_approx := eval(Taylor_chebsum) + b_t·s·ln(s) :
      # numerical expansion for geostrophic flow
> G := add(G||idx, idx = 1 ..4) :

> #Minimise the squared Residual of Taylor's ODE
> Taylor_Residual := simplify(alpha * diff(Taylor_numerical_approx/s, s, s) + beta
      * diff(Taylor_numerical_approx/s, s) - G) :
> Int_Res_t := int( Taylor_Residual^2, s = 0 ..1) :
> Int_Res2_t := collect(Int_Res_t, seq( a_t||i, i = 1 ..N), factor) :
> Int_Res2_t := evalf(minimize(Int_Res2_t, seq( a_t||i, i = 1 ..N), b_t, location)) :

> Solutionarray_t := {op(op(op(Int_Res2_t[2]))[1])} :

> vars_t := [seq( a_t||i,i=1..N),b_t]:

> Solutionarray2_t := evalf(subs(Solutionarray_t, vars_t)) :
```

> # calculate constant so angular momentum is zero

```
> Taylor_numerical_app := Taylor_numerical_approx + s·Cst :
> Taylor_ang_mom := evalf(int(int(int(expand(Taylor_numerical_app * s^2), phi = 0 .. 2 * Pi), z
      = -sqrt(1-s^2) .. sqrt(1-s^2)), s = 0 .. 1)) :
> for i from 1 to N do a_t||i := (Solutionarray2_t[i]) end do:

> b_t := rhs(Solutionarray_t[M]);
```

$$b\_t := -134.765979416910388627129459004664574747470368609064 \quad (53)$$

```
> Taylor_C_value := solve(Taylor_ang_mom = 0):
```

```
> Taylor_numerical_solution := subs(Cst = Taylor_C_value, Taylor_numerical_app) :
> #Plot geostrophic flow solution
> plot(Taylor_numerical_solution, s = 0 ..1);
```