Electronic supplementary materials

to **"An elliptical liquid inclusion in an infinite elastic plane"** by J. Wu, C.Q. Ru and L. Zhang

Can existing solutions of an elastic inclusion directly give the present results by setting the shear modulus to be zero?

It will be showed that even in the simple case with no surface tension, the present results for a liquid inclusion cannot be always easily derived from the existing results of an elastic inclusion by simply setting the shear modulus to be zero and the Poisson's ratio to be 0.5. For example, as showed below, the interface displacement given by the present model for a specific example is considerably different than those obtained from an existing solution of an elastic inclusion.

First, to understand the cause of this difference, it is stated that available known elastic inclusion solutions in plane-elasticity are given in terms of the shear modulus and Poisson's ratio (rather than the bulk modulus). Thus, to make an elastic inclusion equivalent to a given liquid inclusion, one has to set the shear modulus to be zero and, at the same time, the Poisson's ratio has to be 0.5 to ensure a non-zero bulk modulus. In doing so, however, the compressibility and incompressibility of the liquid inclusion is not yet distinguished, and the specific value of bulk modulus of a liquid inclusion is not yet well defined for an elastic inclusion solution.

Second, in existing results [20-22] including reference [22] mentioned by a reviewer, the shear modulus of the inclusion appears in the denominator, for instance, in Eq. (2.13) of reference [20], Eq. (5.19) of reference [21], and Eq. (2.8) of reference [22]. Thus, the existing solutions cannot be directly used for a liquid inclusion. To make the existing solutions available, the shear modulus has to be a small but non-zero number. The existing elastic inclusion model with small shear modulus is actually an approximate model for a liquid inclusion. For instance, the shear stress in the inclusion obtained by this model is small but non-zero under remote shear loading, while the shear stress in static liquids should be zero.

Finally, it will be showed that the existing elastic inclusion model with small shear modulus cannot be always close to the present liquid inclusion model, even in the absence of surface tension. For example, let us calculate the interface displacement of an elastic inclusion based on the known elastic inclusion solutions. From the reference [22] (mentioned by a reviewer) and other references [20, 21] on an elliptical elastic inclusion, the displacements and stresses in the matrix and in the inclusion can be defined by two analytic functions φ and ψ as [23]

$$\begin{aligned}
&\sigma_x^{(j)}+\sigma_y^{(j)}=2\left[\varphi_j'(z)+\overline{\varphi_j'(z)}\right]\\
&\sigma_y^{(j)}-\sigma_x^{(j)}+2i\tau_{xy}^{(j)}=2\left[\bar{z}\varphi_j''(z)+\psi_j'(z)\right]\\
&2G_j(u_x^{(j)}+iu_y^{(j)})=\kappa_j\varphi_j(z)-z\overline{\varphi_j'(z)}-\overline{\psi_j(z)}
\end{aligned} \tag{S1}$$

where, $z = x + iy$; G is the shear modulus, $\kappa = 3-4\nu$ for plane strain, ν is the Poisson's ratio, $j = 1$ indicates the components in the matrix, and $j = 2$ indicates the components in the inclusion. It is well-known that an elliptical elastic inclusion yields a uniform internal stress field under any uniform remote loads at infinity [21], thus the solutions are of the following form

$$\varphi_2 = Lz,\ \psi_2 = Mz \tag{S2}$$

where L and M are complex constants [20]. To match a liquid inclusion at static equilibrium, the internal stress is hydrostatic and uniform. Therefore, from the second equation in (S1), one gets $M \to 0$, as given by Eqs. (5.18)-(5.20) in reference [21] and Eqs. (2.5), (2.13)-(2.17) in reference [20] by setting $G_2 \to 0$, and $\nu_2 \to 0.5$. The value of L and M can be determined by the continuity conditions of displacements and force on the interface of an elastic inclusion [20, 21], as showed in Eqs. (5.18)-(5.20) in [21]. Then from the third equation in (S1) and Eq. (S2), one can calculate the displacements on the interface, as showed in Fig. S1, where the bulk modulus of the liquid inclusion is matched by $K_2 = 2G_2(1+\nu_2)/(3-6\nu_2)$ by taking $G_2 \to 0$ and $\nu_2 \to 0.5$ in a specifically required limiting way. It is seen from Fig. S1 that the existing results of an elastic inclusion are in good agreement with the results obtained by the present liquid inclusion model under uniform remote loads $\sigma_y^0 = 0.2G_1$, $\sigma_x^0 = \tau_{xy}^0 = 0$, but are different than the present results when $\tau_{xy}^0 \neq 0$.

The considerable deviation showed in Fig. S1 is attributed to the fact that the internal shear stress is small but non-zero in the elastic inclusion under remote load $\tau_{xy}^0 \neq 0$. From Eqs. (S1) and (S2) it can be seen that $u_x^{(2)}$ and $u_y^{(2)}$ are linearly related to M/G_2, and $\mathrm{Im}\{M\}/G_2 = \tau_{xy}^{(2)}/G_2$. Thus the non-zero internal shear stress considerably affects the displacement field. Therefore, it is concluded that the results given by the present liquid inclusion model, which are valid for a liquid inclusion, cannot be always easily obtained from the existing solutions of an elastic inclusion by taking shear modulus to be zero.

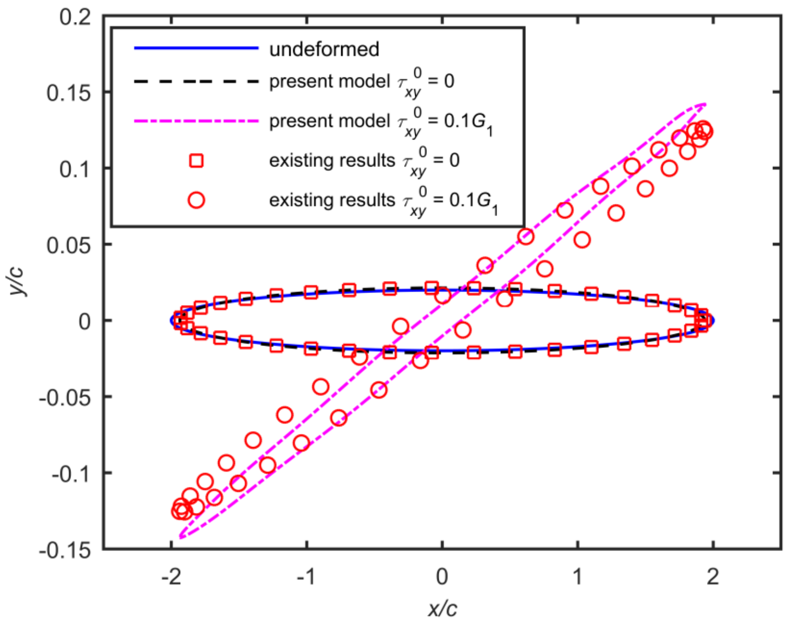

Fig. S1 A comparison on the deformed inclusion shapes given by the present model and the existing elastic inclusion solutions, under uniform remote loads $\sigma_y^0 = 0.2G_1$, $\sigma_x^0 = 0$, and $R = 1.01$, where G_1 is the shear modulus of the matrix, R is a dimensionless parameter defined by the original inclusion shape, and c is the half distance between two foci of the ellipse before deformation. The bulk modulus of the inclusion $K_2 = 5G_1$.