Electronic Supplementary Material - 1 (ESM-1) for "Energy Extraction from Vortex Induced Vibrations using Period-1 Rotation of an Autoparametric Pendulum" Detailed Derivation of the Equations of Motion for the Vortex Induced Vibrations based Energy Harvesters

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1 Vertical Configuration

To derive the EOM for the cylinder coupled with the pendulum corresponding to the vertical configuration, i.e., the cylinder allowed to vibrate along the gravitational direction, we have isolated the system from the wake oscillator and replace interaction with it using an external force S_v as shown in the Fig. 1(i). From Fig. 1(i), the total displacement of the pendulum bob is $(L\sin\theta, -L\cos\theta + Y)$, where Y is the vertical displacement of the cylinder, L is the length of the massless pendulum rod, and θ is the angle made by the pendulum rod with the vertical axis. The kinetic energy of this cylinder-pendulum system can be written as

$$\begin{aligned} \mathrm{KE} &= \frac{M}{2}\left[\left(L\dot{\theta}\cos\theta\right)^2 + \left(L\dot{\theta}\sin\theta + \dot{Y}\right)^2\right] + \frac{1}{2}m_v\dot{Y}^2, \\ &= \frac{M}{2}\left[L^2\dot{\theta}^2 + \dot{Y}^2 + 2L\dot{\theta}\dot{Y}\sin\theta\right] + \frac{1}{2}m_v\dot{Y}^2, \end{aligned} \tag{1}$$

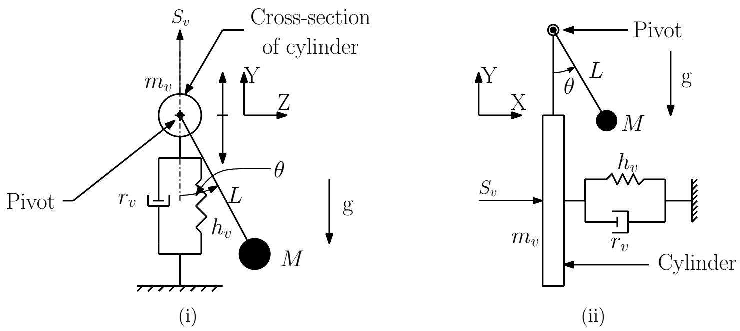

Figure 1: Schematic of energy extraction through VIV. (i) Vertical configuration. (ii) Horizontal case.

where M is the mass of pendulum bob and m_v is the effective mass of the cylinder including the added mass from the fluid. The potential energy of this system is given by

$$\text{PE} = -MgL\cos\theta + \frac{1}{2}h_v Y^2, \tag{2}$$

where h_v is the total effective stiffness of the structure supporting the cylinder. Note that there is a restoring force resulting from the fluid motion as well and h_v includes this stiffness from the fluid as well. We note that the origin of the coordinate system has been taken at the static displacement of the cylinder so that the potential energy due to static displacement is absent in the above equation, i.e., Eq. (2). Having obtained the relevant energies for our system, we can obtain the Lagrangian of our system as

$$\begin{aligned}\mathcal{L} &= \text{KE} - \text{PE}, \\ &= \frac{M}{2}\left[L^2\dot{\theta}^2 + \dot{Y}^2 + 2L\dot{\theta}\dot{Y}\sin\theta\right] + \frac{1}{2}m_v\dot{Y}^2 + MgL\cos\theta - \frac{1}{2}h_v Y^2.\end{aligned} \tag{3}$$

The equation of motion for the cylinder can be obtained using the following Euler-Lagrange equation:

$$\frac{d}{d\bar{t}}\left(\frac{\partial\mathcal{L}}{\partial\dot{Y}}\right) - \frac{\partial\mathcal{L}}{\partial Y} = -r_v\dot{Y} + S_v, \tag{4}$$

where $-r_v\dot{Y}$ is the damping force in the generalized coordinate Y (including dissipation from the fluid) and S_v is the lift force resulting from the alternating vortices (wakes), see reference [1] for more details. From Eq. (3), we have

$$\frac{d}{d\bar{t}}\left(\frac{\partial\mathcal{L}}{\partial\dot{Y}}\right) = \frac{d}{d\bar{t}}\left[(m_v + M)\,\dot{Y} + ML\dot{\theta}\sin\theta\right],$$
$$= (m_v + M)\,\ddot{Y} + ML\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right) \tag{5}$$

and

$$\frac{\partial\mathcal{L}}{\partial Y} = -h_v Y. \tag{6}$$

Substituting the expressions for $\frac{d}{d\bar{t}}\left(\frac{\partial\mathcal{L}}{\partial\dot{Y}}\right)$ and $\frac{\partial\mathcal{L}}{\partial Y}$ in Eq. (4), we get

$$(m_v + M)\,\ddot{Y} + r_v\dot{Y} + h_v Y + ML\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right) = S_v. \tag{7}$$

Note the presence of θ dependent terms in the above equation which account for the fluctuating force on the cylinder resulting from the pendulum motion. Similarly the equation of motion for the pendulum can be obtained as

$$ML^2\ddot{\theta} + C\dot{\theta} + ML(\ddot{Y} + g)\sin\theta = 0\,. \tag{8}$$

To complete our set of EOMs, we require a wake oscillator model for S_v. The wake oscillator equation is directly reproduced from Facchinetti *et al.* [1] with velocity coupling instead of acceleration coupling, which is

$$\ddot{q}_w + \bar{\epsilon}\Omega_f(q_w^2 - 1)\dot{q}_w + \Omega_f^2 q_w = \frac{A_v}{D}\Omega_f\dot{Y}, \tag{9}$$

where $\bar{\epsilon}$ determines the Van der Pol damping strength, Ω_f is the vortex-shedding angular frequency, A_v is nondimensional velocity coupling coefficient and q_w is nondimensional wake variable which is given by $q_w = \frac{2C_L}{C_{L0}}$, where C_L and C_{L0} are vortex lift coefficient and vortex lift coefficient on a fixed structure subjected to vortex shedding, respectively. The lift force S_v can be obtained from the wake variable q_w following the definition of the lift coefficient. We have not provided this expression here as we will be working with the non-dimensionalised equations and hence, prefer to present the final form of S_v in the non-dimensional terms only.

2 Horizontal Configuration

Similarly when the cylinder is restricted to oscillate perpendicular to the gravitational direction, the corresponding isolated system is given by Fig. 1(ii). The total displacement of the pendulum bob is $(L\sin\theta + X, -L\cos\theta)$, where X is the horizontal displacement of the cylinder, L is the length of the pendulum rod and θ is the angle made by the pendulum rod with the vertical axis. The kinetic energy of the cylinder-pendulum system is

$$\begin{aligned}\mathrm{KE} &= \frac{M}{2}\left[\left(L\dot{\theta}\cos\theta + \dot{X}\right)^2 + \left(L\dot{\theta}\sin\theta\right)^2\right] + \frac{1}{2}m_v\dot{X}^2,\\ &= \frac{M}{2}\left[L^2\dot{\theta}^2 + \dot{X}^2 + 2L\dot{\theta}\dot{X}\cos\theta\right] + \frac{1}{2}m_v\dot{X}^2.\end{aligned} \tag{10}$$

The potential energy of our system is given by

$$\mathrm{PE} = -MgL\cos\theta + \frac{1}{2}h_vX^2 \tag{11}$$

which results in the following Lagrangian for the structural system:

$$\begin{aligned}\mathcal{L} &= \mathrm{KE} - \mathrm{PE},\\ &= \frac{M}{2}\left[L^2\dot{\theta}^2 + \dot{X}^2 + 2L\dot{\theta}\dot{X}\cos\theta\right] + \frac{1}{2}m_v\dot{X}^2 + MgL\cos\theta - \frac{1}{2}h_vX^2.\end{aligned} \tag{12}$$

The EOM for the cylinder motion can be obtained from

$$\frac{d}{d\bar{t}}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}}\right) - \frac{\partial\mathcal{L}}{\partial X} = -r_v\dot{X} + S_v, \tag{13}$$

where $-r_v\dot{X}$ is the damping force in the generalized coordinate X and S_v is the lift force due to the wake, see reference [1]. From Eq. (12), we have

$$\begin{aligned}\frac{d}{d\bar{t}}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}}\right) &= \frac{d}{d\bar{t}}\left[(m_v + M)\,\dot{X} + ML\dot{\theta}\cos\theta\right],\\ &= (m_v + M)\,\ddot{X} + ML\left(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta\right).\end{aligned} \tag{14}$$

and

$$\frac{\partial\mathcal{L}}{\partial X} = -h_vX. \tag{15}$$

Substituting the expressions for $\frac{d}{d\bar{t}}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}}\right)$ and $\frac{\partial\mathcal{L}}{\partial X}$ in Eq. (13), we get

$$(m_v + M)\,\ddot{X} + r_v\dot{X} + h_vX + ML\left(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta\right) = S_v\,. \tag{16}$$

Similarly the EOM for the pendulum can be obtained as

$$ML^2\ddot{\theta} + C\dot{\theta} + MgL\sin\theta + ML\ddot{X}\cos\theta = 0\,. \tag{17}$$

The wake oscillator equation with velocity coupling is

$$\ddot{q}_w + \bar{\epsilon}\Omega_f(q_w^2 - 1)\dot{q}_w + \Omega_f^2q_w = \frac{A_v}{D}\Omega_f\dot{X}. \tag{18}$$

References

[1] Facchinetti, M. L., de Langre, E. and Biolley, F., 2004, "Coupling of structure and wake oscillators in vortex-induced vibrations", Journal of Fluids and Structures, **19**(2), pp. 123-140.