Electronic Supplementary Material - 2 (ESM-2) for "Energy Extraction from Vortex Induced Vibrations using Period-1 Rotation of an Autoparametric Pendulum" Numerical Scheme for Continuing Branch of Period-1 Rotation with an Unknown Period for an Autoparametric Pendulum System

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1 Numerical search for period-1 rotation of an unknown period for the autoparametric system

The equation of motion for the full autoparametric system for the vertical configuration is

$$\text{Cylinder Oscillator:} \quad \ddot{y} + \left(2\zeta_v\delta_v + \frac{\gamma_v}{\mu_1}\right)\dot{y} + \delta_v^2 y + \frac{\mu_2}{l_d}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = s_v = M_v q_w, \tag{1}$$

$$\text{Wake Oscillator:} \quad \ddot{q}_w + \bar{\epsilon}(q_w^2 - 1)\dot{q}_w + q_w = A_v\dot{y} \ \text{ and} \tag{2}$$

$$\text{Pendulum Equation:} \quad \ddot{\theta} + c_v\dot{\theta} + (a_v + l_d\ddot{y})\sin\theta = 0, \tag{3}$$

We have three second order ODEs (Eqs. (1), (2) and (3)) which constitute an autonomous system and one among them corresponds to a pendulum for which rotating motions are possible. We want to find initial conditions corresponding to period-1 rotation of an unknown period (determined by the response of the overall system) for the pendulum equation. The solutions corresponding to the other two second order ODEs are periodic in nature when the pendulum undergoes period-1 rotation. As there are three second order ODEs, they can be written as six first order ODEs as follows:

$$\dot{\boldsymbol{y}} + \boldsymbol{f}(\boldsymbol{\alpha}, \boldsymbol{y}) = \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{y} \in \mathbb{R}^6$, $\boldsymbol{f} : \mathbb{R}^6 \to \mathbb{R}^6$ and $\boldsymbol{\alpha}$ is the set of system parameters. The first two variables in $\mathbf{y}$ correspond to displacement and velocity of the cylinder motion, the third and fourth variables correspond to angular displacement and velocity of the pendulum motion, and the fifth and sixth variables correspond to the magnitude and rate of change of the wake variable.

Let the values of the system parameters be such that the pendulum in Eq. (4) is undergoing period-1 rotation of an unknown period. We want to find the initial conditions and time period of this period-1 rotation. As a result there are a total of seven unknowns. We need to get rid of one unknown as the dimension of this system is six. This can be easily done as the system is autonomous in nature. Thus we fix the initial condition corresponding to the sixth variable. We choose $y_6(0) = 0$, where choice of zero on RHS of $y_6(0) = 0$ is arbitrary. This just fixes the phase of the other variables relative to the wake oscillator.

The rotation of pendulum is physically periodic but not mathematically periodic as the angular displacement of pendulum increases/decreases (for anti-clockwise/clockwise rotation) monotonically as time progresses. The angular displacement corresponding to period-1 anti-clockwise rotation advances by 2π in one period. The angular velocity is bounded and periodic even mathematically. Let the initial conditions corresponding to anti-clockwise period-1 rotation be: $y_1(0) = u_1$, $y_2(0) = u_2$, $y_3(0) = u_3$, $y_4(0) = u_4$, $y_5(0) = u_5$ and $y_6(0) = 0$, and the time period is T_p. For this anti-clockwise rotation, we should have $y_1(T_p) = u_1$, $y_2(T_p) = u_2$, $y_3(T_p) = u_3 + 2\pi$, $y_4(T_p) = u_4$, $y_5(T_p) = u_5$ and $y_6(T_p) = 0$. Thus the desired initial conditions (u_1, u_2, u_3, u_4, u_5) and time period (T_p) are required to satisfy the following six equations:

$$y_1(T_p) - u_1 = 0, \tag{5}$$

$$y_2(T_p) - u_2 = 0, \tag{6}$$

$$y_3(T_p) - (u_3 + 2\pi) = 0, \tag{7}$$

$$y_4(T_p) - u_4 = 0, \tag{8}$$

$$y_5(T_p) - u_5 = 0 \text{ and} \tag{9}$$

$$y_6(T_p) = 0, \tag{10}$$

where $y_1(T_p)$, $y_2(T_p)$, $y_3(T_p)$, $y_4(T_p)$, $y_5(T_p)$ and $y_6(T_p)$ all are functions of u_1, u_2, u_3, u_4, u_5 and T_p. Let

$$\mathbf{x} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ T_p \end{bmatrix}.$$

Using the definition of $\mathbf{x}$, we can write Eqs. (5)-(10) as

$$\mathbf{g}(\mathbf{x}) = \mathbf{0}. \tag{11}$$

We can now solve Eq. (11) for $\mathbf{x}$ numerically using the Newton-Raphson method. The evaluation of $\mathbf{g}$ in the Newton-Raphson method requires numerical solution of ODEs, i.e., Eq. (4) in the background. And for continuation purpose we have used a fixed arc-length based continuation scheme described in [1] along with the Newton-Raphson method.

References

[1] Wahi, P., 2005, "A Study of delay differential equations with applications to machine tool vibrations", PhD Thesis, Indian Institute of Science, India.