Supplementary Material

a) Spring-dashpot system and thermodynamics consistency

The mathematical model may also be represented a three-element of nonlinear spring-dashpot system as shown in Figure S1. With the constitutive stress-strain relation, the viscosity $\eta(t)$ (Pa.s) of the dashpot can be derived from this system that satisfies the stress equilibrium as a time-dependent nonlinear function:

$$\eta(t) = \frac{(1+\mu e^{\alpha t/\tau_0})\tau_0 \Delta E}{\mu\alpha e^{\alpha t/\tau_0}} \tag{s1}$$

where μ and α are unitless model parameters, τ_0 is a model parameter (s), and $\Delta E = E_0 - E_\infty$ (Pa).

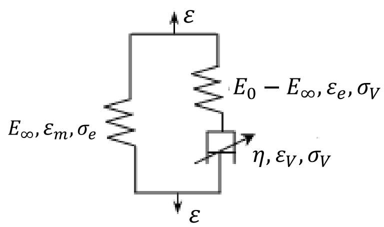


Figure S1. Spring-dashpot system of the model

Some existing models also considered viscosity as a linear or nonlinear function of time [S1]. The nonlinearity may be applied to either the elasticity (ref. [30]) or the viscosity (ref. [31, 32]). The generalized Maxwell model may produce local oscillation for the master curve of loss modulus as contributed primarily by the linear viscosity (ref. [30, 66]). To improving the accuracy of exiting models including the PS as well as for the simplicity purpose with less model parameters, we considered the nonlinearity of the viscosity only. However, when considering the strain hardening for some special materials the nonlinearity is applied to the elasticity-viscosity system.

We have to note that the viscosity value is not measured or validated by experimental testing, instead it is attained through the overall model fitting on the experimental data of $E(t)$ values, the same as being used by other viscoelastic models including the PS for determining the viscosity parameters.

Applying a constant strain ε on the two-element system (see Figure S1), the stress equilibrium satisfies the following:

$$\varepsilon = \varepsilon_e + \varepsilon_V \tag{s2}$$

$$\sigma(t) = E_\infty \varepsilon_m + \Delta E \varepsilon_e \tag{s3}$$

$$\Delta E \varepsilon_e = \eta(t)\dot{\varepsilon}_V(t) \tag{s4}$$

where ε_m is strain of spring E_∞ that $\varepsilon_m = \varepsilon$, ε_e and ε_V are strains of the spring network ΔE and dashpot, respectively.

Substitute equation (s1) and (s2) into (s4):

$$\frac{\mu\alpha e^{\alpha t/\tau_0}}{(1+\mu e^{\alpha t/\tau_0})\tau_0} dt = \frac{d\varepsilon_V(t)}{\varepsilon_e(t)} = \frac{d(\varepsilon - \varepsilon_e(t))}{\varepsilon_e(t)} = -\frac{d\varepsilon_e(t)}{\varepsilon_e(t)} \tag{s5}$$

Then, ε_e can be solved as follows in sequences:

$$\frac{d(\mu e^{\alpha t/\tau_0})}{1+\mu e^{\alpha t/\tau_0}} = -\frac{d\varepsilon_e(t)}{\varepsilon_e(t)} \tag{s6}$$

$$\varepsilon_e(t) = \frac{A}{1+\mu e^{\alpha t/\tau_0}} \tag{s7}$$

where A is a constant. Let A= ε and substitute equation s7 into equation s3 and then divided by ε, the $E(t)$ formula can be derived as:

$$E(t) = \frac{\sigma(t)}{\varepsilon} = E_\infty + \Delta E \varepsilon_e/\varepsilon = E_\infty + \frac{\Delta E}{1+\mu e^{\alpha t/\tau_0}} \tag{s8}$$

By applying the logarithmical scale to both t and $E(t)$, the proposed model formula can be arrived.

The dissipated energy of the viscosity is expressed as [S2]:

$$\mathrm{W} = \eta\dot{\gamma}^2\left(\frac{2\pi}{\omega}\right) \tag{s9}$$

where γ is shear strain, and ω is angular frequency.

The Calusius-Duhem inequality shall be satisfied [S3] for thermodynamic consistency as follows:

$$\rho T\gamma = -\rho\dot{\phi} + \sigma\dot{\varepsilon} - \rho s\dot{T} - \frac{q}{T}\frac{\partial T}{\partial x} \geq 0 \tag{s10}$$

where $\rho T\gamma$ is a specific energy dissipation term, ϕ is the specific free energy, ε is the total strain of the system, and s is the specific entropy. The possibility of the inequality of equation s10 for thermodynamic consistency can be proven in the following procedure.

The total strain can be decomposed into two parts as follows:

$$\varepsilon = \varepsilon_m + \varepsilon_T \tag{s.11}$$

where ε_m is the mechanical strain and ε_T is the thermal strain due to thermal expansion or contraction. The mechanical strain can be further decomposed into two parts as follows (see Figure S1):

$$\varepsilon_m = \varepsilon_e + \varepsilon_v \tag{s12}$$

where ε_e is the elastic strain posed with the network of $E_0 - E_\infty$, and ε_v is the inelastic strain posed by the viscous medium (see Figure S1).

The thermal strain is expressed as follows:

$$\varepsilon_T = \alpha_v(T - T_0) \tag{s13}$$

where α_v is the coefficient of thermal expansion or contraction, and T_0 is the reference temperature. The derivative of the thermal strain can be derived as follows:

$$\dot{\varepsilon}_T = \frac{\partial\alpha_v(T-T_0)}{\partial T}\dot{T} = \alpha_v\dot{T} \tag{s14}$$

The total stress can be decomposed into two parts:

$$\sigma = \sigma_e + \sigma_V \tag{s15}$$

where σ_e is the minimum elastic stress at infinite time posed by E_∞, and σ_V is the stress posed by the elastic network $(E_0 - E_\infty)$ or viscous medium within glass transition (see Figure S1).

The free energy of the system can be given as:

$$\phi = \phi_e(\varepsilon_m, T) + \phi_v(\varepsilon_e, T) + \xi(T) \tag{s16}$$

$\phi_e(\varepsilon_m, T)$ is the energy stored by E_∞, $\phi_v(\varepsilon_\mathrm{e}, T)$ is the energy stored by the elastic network $(E_0 - E_\infty)$ and viscous medium, and $\xi(T)$ is energy related to the heat capacity.

Substitute equation (s11) and (s16) into equation (s10) to reach the following equilibrium:

$$\rho T\gamma = -\rho\left(\dot{\phi}_e(\varepsilon_m, T) + \dot{\phi}_v(\varepsilon_\mathrm{e}, T) + \dot{\xi}(T)\right) + \sigma(\dot{\varepsilon}_m + \dot{\varepsilon}_T) - \rho s\dot{T} - \frac{q}{T}\frac{\partial T}{\partial x} \tag{s17}$$

Substitute equation (s10) to (s15) into equation (s17), and apply the chain rule to achieve the following:

$$\rho T\gamma = -\rho\left(\frac{\partial\phi_e}{\partial\varepsilon_m}\dot{\varepsilon}_m + \frac{\partial\phi_e}{\partial T}\dot{T} + \frac{\partial\phi_v}{\partial\varepsilon_\mathrm{e}}\dot{\varepsilon}_\mathrm{e} + \frac{\partial\phi_v}{\partial T}\dot{T} + \frac{\partial\varepsilon}{\partial T}\dot{T}\right)$$
$$+\left(\sigma_e\dot{\varepsilon}_m + \sigma_\mathrm{V}(\dot{\varepsilon}_\mathrm{e} + \dot{\varepsilon}_\mathrm{v}) + \sigma\alpha_v\dot{T}\right) - \rho s\dot{T} - \frac{q}{T}\frac{\partial T}{\partial x} \tag{s18}$$

This can ber rearranged as follows:

$$\rho T\gamma = \left(\sigma_e - \rho\frac{\partial\phi_e}{\partial\varepsilon_m}\right)\dot{\varepsilon}_m + \left(\sigma_\mathrm{V} - \rho\frac{\partial\phi_v}{\partial\varepsilon_\mathrm{e}}\right)\dot{\varepsilon}_\mathrm{e}$$
$$+\left[\alpha_v\sigma + \frac{\partial\phi_e}{\partial T} - \rho\left(s + \frac{\partial\phi_v}{\partial T}\right) + \frac{\partial\varepsilon}{\partial T}\right]\dot{T} + \sigma_\mathrm{V}\dot{\varepsilon}_\mathrm{V} - \frac{q}{T}\frac{\partial T}{\partial x} \tag{s19}$$

To satisfy $\rho T\gamma \geq 0\ \forall\ \dot{\varepsilon}_m, \dot{\varepsilon}_e, \dot{T}$, the coefficients of the ter14ms $(\dot{\varepsilon}_m, \dot{\varepsilon}_e, \dot{T})$ have to vanish as follows:

$$\sigma_e - \rho\frac{\partial\phi_e}{\partial\varepsilon_m} = 0 \tag{s20}$$

$$\sigma_\mathrm{V} - \rho\frac{\partial\phi_v}{\partial\varepsilon_\mathrm{e}} = 0 \tag{s21}$$

$$\alpha_v\sigma + \frac{\partial\phi_e}{\partial T} - \rho\left(s + \frac{\partial\phi_v}{\partial T}\right) + \frac{\partial\varepsilon}{\partial T} = 0 \tag{s22}$$

According to the stress equilibrium, it can be true that the viscous medium poses the same amount of stress as that of the elastic network during the glass transition $(E_\infty < E(t) < E_0)$. Thus, σ_V can be expressed as a stress of the viscous medium as follows:

$$\sigma_\mathrm{v} = \dot{\varepsilon}_\mathrm{v}\eta \tag{s23}$$

where η is viscosity. According to Fourier's rule, the heat flux is expressed as follows:

$$q = -\gamma\frac{\partial T}{\partial x} \tag{s24}$$

where γ is thermal conductivity. The next step is to substitute equation (s10) to (s14) into equation (s10):

$$\rho T\gamma = \sigma_\mathrm{v}\dot{\varepsilon}_\mathrm{v} - \frac{q}{T}\frac{\partial T}{\partial x} = \frac{\sigma_\mathrm{v}^2}{\eta} + \gamma\left(\frac{\partial T}{\partial x}\right)^2 \geq 0 \tag{s25}$$

Therefore, $\rho T\gamma \geq 0$.

b) Optimization for fitting model parameters

Both the proposed model formula and PS are of C^∞ class or smooth function, e.g. for the PS the k^{th} order derivative exists for $k \to \infty$ such that $E^k(t) = \sum_{i=1}^{N}(-1)^k E_i^{k+1}\eta_i \exp(-E_i/\eta_i t)$. Here we evaluated these two models for curve fitting. We used the nonlinear reduced gradient method embedded in the Microsoft Excel Solver, one of the most popular optimization methods, to fit model parameters by minimizing the objective function:

$$\min f(x_1, x_2, \ldots., x_m)\ \ \forall\, x_i > 0 \tag{s27}$$

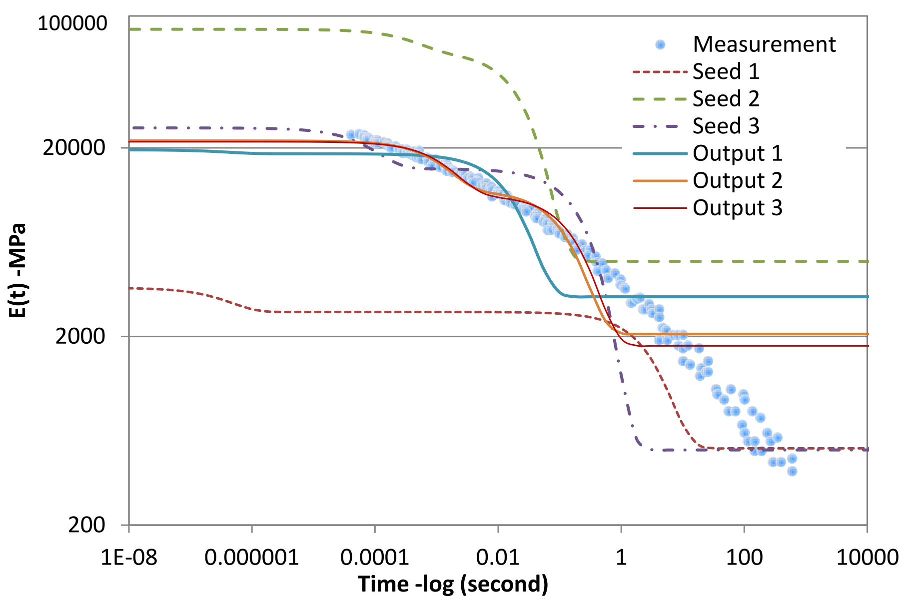
$$f(x_1, x_2, \ldots, x_m) = \sum_i^N \left[E_i(t, x_1, x_2, \ldots, x_m) - \hat{E}_i(t)\right]^2 \tag{s28}$$

where f is the objective function, $E_i(t, x_1, x_2, \ldots, x_m)$ and $\hat{E}_i(t)$ is the predicted and measured data point, respectively, and x_{1-m} is the model parameters.

For this optimization method, the gradients (derivatives of f with respect to each model parameter) are calculated using the central finite difference (CFD) method. The iteration goal is to satisfy the first order essential optimal condition (i.e. the gradient equals or is close to zero).

Figure S2 illustrates the optimization results for the PS with $n = 2$ and the proposed model with the same number of model parameters. Three groups of seed values (the initial guessed model parameters as inputs) were evaluated to consider three general cases of optimization outputs: 1) fitted modulus valus are higher than, 2) lower than, and 3) close to measurment values. Results show that the PS has produced varaible results (i.e. different fittign curve shapes) when using variable seed values (see S2a). However, the proposed model yields more stable and unique solution for these three seed groups, indicating its higher stability for convergence (see Figure S2b). When using a large term number n with more model parameters for the PS, its variability could be greater since it becomes more difficult to estimate proper seed values. In comparison, for the porposed model it is more easy to estimate seed values for model parameters. For example, its E_∞ and E_0 values may be estimated according to the normal range of modulus valuies for a specific material (i.e. the minimum and maximum value is relatively close to E_∞ and E_0, respectively).

a)



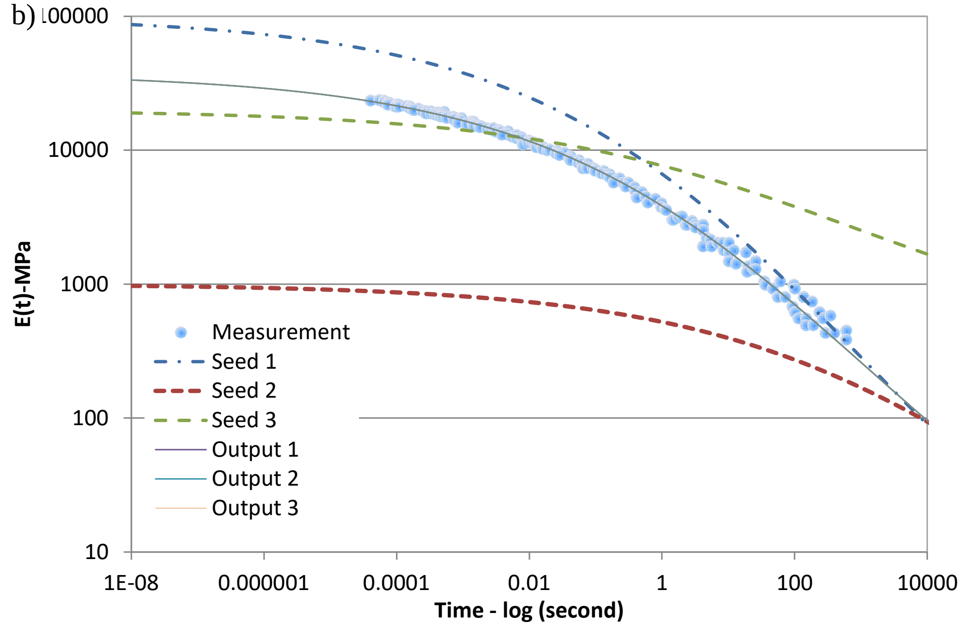


Figure S2. Optimization analysis to fit experimental data: a) Prony series and b) proposed model both with three seed inputs and the proposed one shows more robust and accurate convergence than PS model.

c) Model data fitting and prediction template

Please see supplementary Microsoft Excel 2010 file "Model_Fit_Prediction_1DStressStrain_Analysis.xlsm" for model fitting, prediction, and 1-D stress-strain analysis as a template.

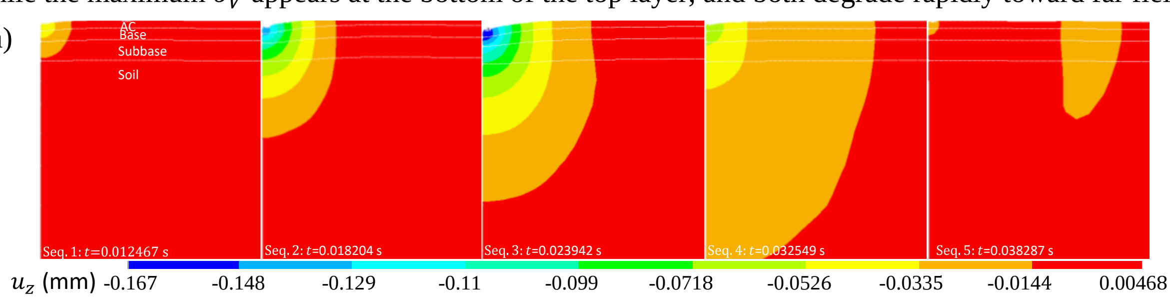
d) Computer code

Please see supplementary computer code.txt files: 1) 1-D_stress_strain_calculation_VBA_code.txt, and 2) Axisymmetric_model_VEstiffness_matrix_Fortran_subroutines.txt

e) Simulated viscoelastic response of multilayer pavement structure

Figure S**3**. presents simulated responses of displacement and stresses with space and time. Displacement waves propagate to far fields with time (see **Figure** S3a). Shear stress τ_{rz} (**Figure** S**3**b) and von Mises stress σ_V (**Figure** S3c) have concentrated within the top viscoelastic AC layer, which is primarily due to its much higher modulus than the underlying layers. The maximum τ_{zr} occurs at the mid while the maximum σ_V appears at the bottom of the top layer, and both degrade rapidly toward far fields.

a)



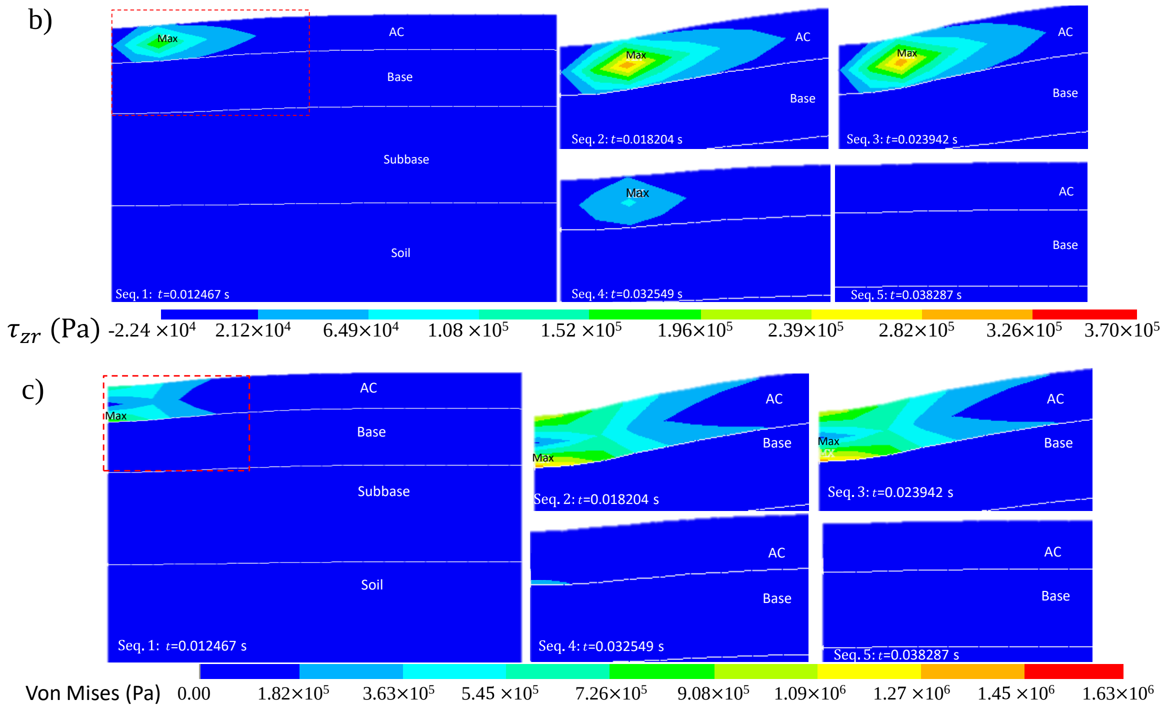


Figure S3. Simulated responses of the layered structure: a) vertical displacement on top showing wave propagation toward far fields; b) shear stress; and c) von Mises stress concentrated within the top layer.

Reference

[S1] A.E. Vardy and J.M.B. Brown. 2011 Laminar pipe flow with time-dependent viscosity. Journal of Hydroinformatics 9, 729-740.

[S2] Vilastic Scientific. Rheological parameters for viscoelastic materials. http://www.vilastic.com/tech3.htm (2000)

[S3] P. Haupt. 1993 Non-equilibrium thermodynamics with applications to solids, in “CISM-Course 336” edited by W. Muschik, Springer Verlag, New York.